Lecture 25

Main Points:

- Proof of Curry Schoenberg Theorem

Writing B-splines as sums of shifted power functions

We showed that it is possible to write a B-spline function $\mathcal{B}_i^d(t)$ in terms of shifted power functions, specifically those which are indicated by the subsequence $t_i, \ldots, t_{i+d+1}$ which is used to define the B-spline.

For instance, in the case where the knots $t_i, \ldots, t_{i+d+1}$ are all simple (of multiplicity one), we can write $\mathcal{B}_i^d(t)$ as:

$$\begin{aligned}
\mathcal{B}_i^d(t) &= (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, \ldots, t_{i+d+1}](t-x)_+^d \\
&= (-1)^{d+1}(t_{i+d+1} - t_i)\frac{1}{D}\begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & (t-t_i)_+^d \\ 1 & t_{i+1} & t_{i+1}^2 & \cdots & t_{i+1}^{d-1} & (t-t_{i+1})_+^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & (t-t_{i+d+1})_+^d \end{vmatrix}
\end{aligned}$$

where D is the Vandermonde determinant:

$$D = \begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & t_i^d \\ 1 & t_{i+1} & t_{i+1}^2 & \cdots & t_{i+1}^{d-1} & t_{i+1}^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & t_{i+d+1}^d \end{vmatrix}.$$

By performing a cofactor expansion of the previous determinant along the last column, we can write the B-spline as a sum:

$$\mathcal{B}_i^d(t) = a_i(t-t_i)_+^d + a_{i+1}(t-t_{i+1})_+^d + \cdots + a_{i+d+1}(t-t_{i+d+1})_+^d.$$

To allow for multiplicities higher than one, we could have for instance knots $t_i = t_{i+1} < t_{i+2} < \cdots < t_{i+d+1}$. In this case we have:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, \ldots, t_{i+d+1}](t-x)_+^d$$

$$= \quad (-1)^{d+1}(t_{i+d+1}-t_i)\frac{1}{D}\begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & (t-t_i)_+^d \\ 0 & 1 & 2t_i & \cdots & (d-1)t_{i+1}^{d-2} & -d(t-t_{i+1})_+^{d-1} \\ 1 & t_{i+2} & t_{i+2}^2 & \cdots & t_{i+2}^{d-1} & (t-t_{i+2})_+^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & (t-t_{i+d+1})_+^d \end{vmatrix}$$

where D is now the Confluent Vandermonde determinant:

$$D = \begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & t_i^d \\ 0 & 1 & 2t_i & \cdots & (d-1)t_{i+1}^{d-2} & dt_{i+1}^{d-1} \\ 1 & t_{i+2} & t_{i+2}^2 & \cdots & t_{i+2}^{d-1} & t_{i+2}^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & t_{i+d+1}^d \end{vmatrix}.$$

Again by performing a cofactor expansion of the previous determinant along the last column, we can write the B-spline as a sum:

$$\mathcal{B}_i^d(t) = a_i(t-t_i)_+^d + a_{i+1}(t-t_i)_+^{d-1} + a_{i+2}(t-t_{i+2})_+^d + \cdots + a_{i+d+1}(t-t_{i+d+1})_+^d.$$

Note that in this case the sum involves one shifted power function of degree $d-1$, corresponding to the second row in the determinant which gives the B-spline coefficients.

Writing a shifted power basis with increasing degree order

It is an important (but trivial) point, to note that we can write a shifted power function basis with increasing degrees. Of course, we can write a basis in any order since it is a set of functions and is independent of order. However, the order can sometimes be a crucial point, especially when the goal is to find a simple change of basis matrix. This is precisely why we choose to write the shifted power basis with increasing degree order. We will see that the matrix of coordinate vectors of the B-splines, with respect to the shifted power basis, is then lower triangular with nonzero diagonal entries, and thus has nonzero determinant. This matrix then gives the change of basis from B-splines to shifted power basis.

Examples:

- Let $V = P_{2,\mathbf{r}}^3[0,1,2,3,4]$, with $\mathbf{r} = (1,0,1)$. The multiplicity sequence is then $\mathbf{m} = (1,2,1)$. Then a shifted power basis can be written in terms of the knot sequence: $\mathbf{t} = \{0,0,0,1,2,2,3\}$. The basis, in degree-increasing order, is then:
$$\{(t-0)_+^0, (t-0)_+^1, (t-0)_+^2, (t-1)_+^2, (t-2)_+^1, (t-2)_+^2, (t-3)_+^2\}.$$
- Let $V = P_{3,\mathbf{r}}^4[0,1,2,3,4]$, with $\mathbf{r} = (2,2,1)$. The multiplicity sequence is then $\mathbf{m} = (1,1,2)$. Then a shifted power basis can be written in terms of the knot sequence: $\mathbf{t} = \{0,0,0,0,1,2,3,3,4,4,4,4\}$. The basis, in degree-increasing order, is then:
$$\{(t-0)_+^0, (t-0)_+^1, (t-0)_+^2, (t-0)_+^3, (t-1)_+^3, (t-2)_+^3, (t-3)_+^2, (t-3)_+^3\}.$$

Proof of Curry Schoenberg Theorem (B-spline basis theorem)

As indicated in the above paragraph, it is enough to show that the matrix of coordinate vectors of B-splines has nonzero determinant.

Examples:

- Let $V = P^3_{2,\mathbf{r}}[0,1,2,3,4]$, with $\mathbf{r} = (1,0,1)$. The shifted power basis was written above. We can also write a knot sequence whose associated B-splines are a basis of V. Such a knot sequence is: $\mathbf{t}' = \{0,0,0,1,2,2,3,4,4,4\}$. The B-splines of degree 2 associated to $\mathbf{t}'$ are:

$$\mathcal{B}_2(\mathbf{t}') = \{\mathcal{B}^2_0(t), \mathcal{B}^2_1(t), \mathcal{B}^2_2(t), \mathcal{B}^2_3(t), \mathcal{B}^2_4(t), \mathcal{B}^2_5(t), \mathcal{B}^2_6(t)\}.$$

  Each of these B-splines can be written in terms of the shifted power basis as follows. First we compute the relevant confluent Vandermonde determinants:

$$D_0 = D(0,0,0,1) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 0 & 1 & 2\cdot 0 & 3\cdot 0^2 \\ 0 & 0 & 2 & 6\cdot 0 \\ 1 & 1 & 1^2 & 1^3 \end{vmatrix} = 2, \qquad D_1 = D(0,0,1,2) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 0 & 1 & 2\cdot 0 & 3\cdot 0^2 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \end{vmatrix} = 4,$$

$$D_2 = D(0,1,2,2) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \end{vmatrix} = 4, \qquad D_3 = D(1,2,2,3) = \begin{vmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \\ 1 & 3 & 3^2 & 3^3 \end{vmatrix} = 2,$$

$$D_4 = D(2,2,3,4) = \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 4, \qquad D_5 = D(2,3,4,4) = \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 0 & 1 & 2\cdot 4 & 3\cdot 4^2 \end{vmatrix} = 4,$$

$$D_6 = D(3,4,4,4) = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 0 & 1 & 2\cdot 4 & 3\cdot 4^2 \\ 0 & 0 & 2 & 6\cdot 4 \end{vmatrix} = 2.$$

  Next, we compute the B-splines as sums of shifted power functions. Since each of these can be easily done in a symbolic algebra package, such as PARI, we include the PARI commands which show the input and output. The determinant always has last column with symbols `a,b,c,d` instead of the shifted power functions, for ease of input. The coefficients are then assigned to the correct functions in the output. We also give the coordinate vector of the B-spline with respect to the degree-increasing ordered basis:

$$\{(t-0)^0_+, (t-0)^1_+, (t-0)^2_+, (t-1)^2_+, (t-2)^1_+, (t-2)^2_+, (t-3)^2_+\}.$$

$$\begin{aligned}
\mathcal{B}^2_0(t) &= (-1)^{2+1}(1-0)[0,0,0,1](t-x)^2_+ \\
&= \frac{-1}{D_0}\begin{vmatrix} 1 & 0 & 0^2 & (t-0)^2_+ \\ 0 & 1 & 2\cdot 0 & -2(t-0)^1_+ \\ 0 & 0 & 2 & 2(t-0)^0_+ \\ 1 & 1 & 1^2 & (t-1)^2_+ \end{vmatrix} \\
&= (t-0)^2_+ - 2(t-0)^1_+ + (t-0)^0_+ - (t-1)^1_+.
\end{aligned}$$

PARI input: $(-1/2)*\mathtt{matdet}([1,0,0,\mathtt{a};0,1,0,-2*\mathtt{b};0,0,2,2*\mathtt{c};1,1,1,\mathtt{d}])$

PARI output: $\mathtt{a}-2*\mathtt{b}+\mathtt{c}-\mathtt{d}$

Coordinate vector: $(1,-2,1,-1,0,0,0)$

$$\begin{aligned}
\mathcal{B}_1^2(t) &= (-1)^{2+1}(1-0)[0,0,1,2](t-x)_+^2 \\
&= \frac{-1}{D_1}\begin{vmatrix} 1 & 0 & 0^2 & (t-0)_+^2 \\ 0 & 1 & 2\cdot 0 & -2(t-0)_+^1 \\ 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \end{vmatrix} \\
&= -\frac{3}{4}(t-0)_+^2+(t-0)_+^1+(t-1)_+^2-\frac{1}{4}(t-2)_+^1.
\end{aligned}$$

PARI input: $(-1/4)*\mathtt{matdet}([1,0,0,\mathtt{a};0,1,0,-2*\mathtt{b};1,1,1,\mathtt{c};1,2,4,\mathtt{d}])$

PARI output: $-3/4*\mathtt{a}+\mathtt{b}+\mathtt{c}-1/4*\mathtt{d}$

Coordinate vector: $(0,1,-\frac{3}{4},1,-\frac{1}{4},0,0)$

$$\begin{aligned}
\mathcal{B}_2^2(t) &= (-1)^{2+1}(1-0)[0,1,2,2](t-x)_+^2 \\
&= \frac{-1}{D_2}\begin{vmatrix} 1 & 0 & 0^2 & (t-0)_+^2 \\ 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \end{vmatrix} \\
&= \frac{1}{4}(t-0)_+^2-(t-1)_+^2+\frac{3}{4}(t-2)_+^2+(t-2)_+^1.
\end{aligned}$$

PARI input: $(-1/4)*\mathtt{matdet}([1,0,0,\mathtt{a};1,1,1,\mathtt{b};1,2,4,\mathtt{c};0,1,4,-2*\mathtt{d}])$

PARI output: $1/4*\mathtt{a}-\mathtt{b}+3/4*\mathtt{c}+\mathtt{d}$

Coordinate vector: $(0,0,\frac{1}{4},-1,1,\frac{3}{4},0)$

$$\begin{aligned}
\mathcal{B}_3^2(t) &= (-1)^{2+1}(1-0)[1,2,2,3](t-x)_+^2 \\
&= \frac{-1}{D_3}\begin{vmatrix} 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \\ 1 & 3 & 3^2 & (t-3)_+^2 \end{vmatrix} \\
&= \frac{1}{2}(t-1)_+^2+0\cdot(t-2)_+^2-2(t-2)_+^1-\frac{1}{2}(t-3)_+^2.
\end{aligned}$$

PARI input: $(-1/2)*\mathtt{matdet}([1,1,1,\mathtt{a};1,2,4,\mathtt{b};0,1,4,-2*\mathtt{c};1,3,9,\mathtt{d}])$

PARI output: $1/2*\mathtt{a}-2*\mathtt{c}-1/2*\mathtt{d}$

Coordinate vector: $(0,0,0,\frac{1}{2},-2,0,-1)$

$$\begin{aligned}
\mathcal{B}_4^2(t) &= (-1)^{2+1}(1-0)[2,2,3,4](t-x)_+^2 \\
&= \frac{-1}{D_3}\begin{vmatrix} 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \\ 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \end{vmatrix} \\
&= -\frac{5}{4}(t-2)_+^2 + (t-2)_+^1 + 2(t-3)_+^2 - \frac{3}{4}(t-4)_+^2.
\end{aligned}$$

`PARI input:` $(-1/4)*\texttt{matdet}([1,2,4,\texttt{a};0,1,2,-2*\texttt{b};1,3,9,\texttt{c};1,4,16,\texttt{d}])$

`PARI output:` $-5/4*\texttt{a}+\texttt{b}+2*\texttt{c}-3/4*\texttt{d}$

Coordinate vector: $(0,0,0,0,1,-\frac{5}{4},2)$

$$\begin{aligned}
\mathcal{B}_5^2(t) &= (-1)^{2+1}(1-0)[2,3,4,4](t-x)_+^2 \\
&= \frac{-1}{D_5}\begin{vmatrix} 1 & 2 & 2^2 & (t-2)_+^2 \\ 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \\ 0 & 1 & 2\cdot 4 & -2(t-4)_+^1 \end{vmatrix} \\
&= \frac{1}{4}(t-2)_+^2 - (t-3)_+^2 + \frac{3}{4}(t-4)_+^2 + (t-4)_+^1.
\end{aligned}$$

`PARI input:` $(-1/4)*\texttt{matdet}([1,2,4,\texttt{a};1,3,9,\texttt{b};1,4,16,\texttt{c};0,1,8,-2*\texttt{d}])$

`PARI output:` $1/4*\texttt{a}-\texttt{b}+3/4*\texttt{c}+\texttt{d}$

Coordinate vector: $(0,0,0,0,0,\frac{1}{4},-1)$

$$\begin{aligned}
\mathcal{B}_6^2(t) &= (-1)^{2+1}(1-0)[3,4,4,4](t-x)_+^2 \\
&= \frac{-1}{D_6}\begin{vmatrix} 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \\ 0 & 1 & 2\cdot 4 & -2(t-4)_+^1 \\ 0 & 0 & 2 & 2(t-4)_+^0 \end{vmatrix} \\
&= (t-3)_+^2 - (t-4)_+^2 - 2(t-4)_+^1 - (t-4)_+^0.
\end{aligned}$$

`PARI input:` $(-1/2)*\texttt{matdet}([1,3,9,\texttt{a};1,4,16,\texttt{b};0,1,8,-2*\texttt{c};0,0,2,2*\texttt{d}])$

`PARI output:` $\texttt{a}-\texttt{b}-2*\texttt{c}-\texttt{d}$

Coordinate vector: $(0,0,0,0,0,0,1)$

Finally, we can write the matrix of coordinate vectors and see that it indeed is lower triangular, with nonzero diagonal entries, hence has nonzero determinant. This matrix, called A below, is then the change of basis matrix from the B-spline to the shifted power basis, with respect to the chosen knot sequences $\mathbf{t}$ and $\mathbf{t}'$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & -\frac{5}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 1 \end{pmatrix}$$

The inverse of A is then the change of basis matrix from the shifted power basis, with knot sequence $\mathbf{t}$, to the B-spline basis, with knot sequence $\mathbf{t}'$.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 & 0 \\ 2 & 4 & 8 & 2 & 0 & 0 & 0 \\ \frac{5}{2} & \frac{21}{4} & \frac{1}{2} & 4 & 1 & 0 & 0 \\ \frac{13}{2} & \frac{69}{4} & 48 & 20 & 5 & 4 & 0 \\ \frac{7}{2} & \frac{43}{4} & 32 & 14 & 3 & 4 & 1 \end{pmatrix}$$

- Let $V = P^5_{3,\mathbf{r}}[0,1,2,3,4,5]$, with $\mathbf{r} = (0,1,2,1)$. The associated multiplicity sequence is then: $\mathbf{m} = (3,2,1,2)$. A shifted power basis can be written with knot sequence: $\mathbf{t} = \{0,0,0,0,1,1,1,2,2,3,4,4\}$. The shifted power basis $sp_3(\mathbf{t})$, written in degree-increasing order, is then:

$$\{(t-0)^0_+, (t-0)^1_+, (t-0)^2_+, (t-0)^3_+, (t-1)^1_+, (t-1)^2_+, (t-1)^3_+, (t-2)^2_+, \\ (t-2)^3_+, (t-3)^3_+, (t-4)^2_+, (t-4)^3_+\}.$$

We can also write a knot sequence whose associated B-splines are a basis of V. Such a knot sequence is: $\mathbf{t}' = \{0,0,0,0,1,1,1,2,2,3,4,4,5,5,5,5\}$. The B-splines of degree 3 associated to $\mathbf{t}'$ are:

$$\mathcal{B}_3(\mathbf{t}') = \{\mathcal{B}^3_0(t), \mathcal{B}^3_1(t), \ldots, \mathcal{B}^3_{10}(t), \mathcal{B}^3_{11}(t)\}.$$

Without working out the numerical values of the shifted power basis coordinate vectors for each B-spline, we can still see the general shape of the change of basis matrix, called B below, and verify that indeed it has nonzero determinant. This follows from the fact that for each B-spline, the coefficient of the first shifted power function is always nonzero. This is due to the fact that the coefficient comes from a confluent Vandermonde determinant.

In the change of basis matrix B below, the stars represent nonzero values which are computed from confluent Vandermonde determinants, and the dots represent the other coefficients which contribute to the B-spline and which may or may not be zero. This illustrates the proof of the Curry-Schoenberg Theorem, which is simply the claim that the matrix has nonzero determinant.

The vector to the left shows the shifted power basis which corresponds to the columns of the matrix.

$$
\mathbf{v} = \begin{pmatrix} (t-0)_+^0 \\ (t-0)_+^1 \\ (t-0)_+^2 \\ (t-0)_+^3 \\ (t-1)_+^1 \\ (t-1)_+^2 \\ (t-1)_+^3 \\ (t-2)_+^2 \\ (t-2)_+^3 \\ (t-3)_+^3 \\ (t-4)_+^2 \\ (t-4)_+^3 \end{pmatrix}
\qquad
B = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & . & . & . & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & . & . & . & . & . & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & . & . & . & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & . & . & . & . & *
\end{pmatrix}
$$