Lecture 3

Main Points:

- Proofs of basis properties of polynomials

Linear independence for polynomials

To prove that a set of polynomials is linearly independent, we refer to the coordinate vectors with respect to a basis, usually the standard basis. If the set of coordinate vectors is linearly independent, then the corresponding polynomials are also independent. (This follows from the important theorem from linear algebra from lecture 1.) In order to check for linear independence of coordinate vectors, we use the determinant criterion:

Determinant Criterion for linear independence of column vectors:

A set of n column vectors in R^n is linearly independent if and only if the determinant of the matrix of column vectors is nonzero. (Note: the columns can be in any order in the matrix.)

Examples:

- The **shifted basis** $\{1, t-c, (t-c)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & -c & c^2 \\ 0 & 1 & -2c \\ 0 & 0 & 1 \end{pmatrix}$$
which has determinant 1 (equal to the product of diagonal entries since the matrix is upper triangular.)
- The **Bernstein basis** $\{(1-t)^2, 2t(1-t), t^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$
which has determinant 2 (equal to the product of diagonal entries since the matrix is lower triangular.)
- The **Vandermonde basis** $\{(t-1)^2, (t-2)^2, (t-3)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix}$$
which has determinant 2 (which can be computed with the method of cofactor expansion.)
- The **top-down basis** $\{(t-1)^2, t-1, (t-2)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & -1 & 4 \\ -2 & 1 & -4 \\ 1 & 0 & 1 \end{pmatrix}$$
which has determinant -1 (which can be computed with the method of cofactor expansion.)

In order to do higher degree cases of shifted or Bernstein bases, the matrices are still triangular and so the determinants follow the same pattern.

It is useful to recall the binomial expansion:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2 b^{n-2} + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

Examples:

- The general shifted basis $\{1, t-c, (t-c)^2, \ldots, (t-c)^d\}$ has upper triangular change of basis matrix (to the standard basis), which has determinant 1:

$$\begin{vmatrix} 1 & -c & c^2 & \cdots & (-c)^d \\ 0 & 1 & -2c & \cdots & \binom{d}{d-1}(-c)^{d-1} \\ 0 & 0 & 1 & \cdots & \binom{d}{d-2}(-c)^{d-2} \\ 0 & 0 & 0 & \cdots & \binom{d}{d-3}(-c)^{d-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1.$$

- The general Bernstein basis

$$\{B_0^d(t), B_1^d(t), \ldots, B_d^d(t)\} = \{(1-t)^d, d(1-t)^{d-1}t, \binom{d}{2}(1-t)^{d-2}t^2, \ldots, t^d\}$$

  has lower triangular change of basis matrix (to the standard basis), which has nonzero determinant (since all the binomial coefficients are nonzero):

$$\begin{vmatrix} \binom{d}{0} & 0 & 0 & \cdots & 0 \\ * & \binom{d}{1} & 0 & \cdots & 0 \\ * & * & \binom{d}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & \binom{d}{d} \end{vmatrix} = \binom{d}{0}\binom{d}{1}\binom{d}{2}\cdots\binom{d}{d} \neq 0.$$

For Vandermonde and top-down bases, however, we need different formulas. We start with the Vandermonde determinant.

Vandermonde Determinant Formula:

$$\begin{vmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix} = \prod_{0 \le i < j \le d} (t_j - t_i)$$

Examples:

- A Vandermonde determinant with $t_0 = 2$, $t_1 = 3$, $t_2 = 4$:

$$\begin{vmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{vmatrix} = (3-2)(4-2)(4-3) = 2$$

- A Vandermonde determinant with $t_0 = -2$, $t_1 = 5$, $t_2 = 3$:

$$\begin{vmatrix} 1 & -2 & 2^2 \\ 1 & 5 & 3^2 \\ 1 & 3 & 4^2 \end{vmatrix} = (5-(-2))(3-(-2))(3-5) = -70$$

Proof of Vandermonde Determinant Formula:

Let

$$D = \begin{vmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix}.$$

This is an $n \times n$ determinant, with $n = d+1$. To prove this formula, we will use an induction argument. The induction hypothsis is that the formula is true for $n \le d$. The base case is $d = 0$. We can check this case immediately, since the formula then simply says that $1 = 1$. The case $d = 1$ can also be checked directly, which is:

$$D = \begin{vmatrix} 1 & t_0 \\ 1 & t_1 \end{vmatrix} = t_1 - t_0$$

which we know to be true from the 2×2 determinant formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So we want to show that $D = \prod_{0 \le i < j \le d} (t_j - t_i)$. In order to show this, we introduce a polynomial which is defined as a determinant:

$$P(x) = \begin{vmatrix} 1 & x & x^2 & \cdots & x^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix}.$$

Because we can expand by cofactors along the top row of the determinant, it is clear that $P(x)$ is a polynomial of degree at most d, with coefficients given by determinants using the values $t_1, \ldots, t_d$. From the definitions, we also have $D = P(t_0)$. Next, we will factor the polynomial $P(x)$ in order to arrive at the determinant formula.

To factor $P(x)$ we simply notice that it must have zeros given by $t_1, \ldots, t_d$. This follows from the property of a determinant which says that if two rows or two columns are identical, then the determinant equals zero. Each of these zeros corresponds to a factor and so we have d factors:

$$P(x) = C \cdot (x - t_1)(x - t_2) \cdots (x - t_d),$$

where C is some constant. Since this polynomial has degree d, there can be no other factors. We can also determine the constant C by looking at the original definition of $P(x)$ using the determinant. This tells us that the coefficient of x^d must be a cofactor determinant given by:

$$\begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{d-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^{d-1} \end{vmatrix}.$$

This has the same form as the original determinant, so we can use the induction step: We assume that the original formula works for all smaller Vandermonde determinants, such as this one. This says that the coefficient C must be:

$$C = \prod_{1 \le i < j \le d} (t_j - t_i)$$

But then

$$\begin{aligned} D = P(t_0) &= C \cdot (t_0 - t_1)(t_0 - t_2) \cdots (t_0 - t_d) && (1) \\ &= \left(\prod_{1 \le i < j \le d} (t_j - t_i) \right) \cdot (t_0 - t_1)(t_0 - t_2) \cdots (t_0 - t_d) && (2) \\ &= \prod_{0 \le i < j \le d} (t_j - t_i) && (3) \end{aligned}$$

which completes the proof.

Examples:

- To see how this proof applies to a Vandermonde basis of degree 2 we can start with the basis:

$$\{(t-t_0)^2, (t-t_1)^2, (t-t_2)^2\}.$$

  Now we would like to show that these polynomials are linearly independent, so we first write down the matrix of column vectors:

$$\begin{pmatrix} t_0^2 & t_1^2 & t_2^2 \\ -2t_0 & -2t_1 & -2t_2 \\ 1 & 1 & 1 \end{pmatrix}$$

  We can apply a few properties of determinants to see that this is a constant times a Vandermonde determinant. Call this matrix A. Then by swapping the first and last rows, then factoring out a -2 from the second row, and taking a transpose, and applying the Vandermonde formula yields:

$$det(A) = -\begin{vmatrix} 1 & 1 & 1 \\ -2t_0 & -2t_1 & -2t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{vmatrix} = -(-2)\begin{vmatrix} 1 & 1 & 1 \\ t_0 & t_1 & t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{vmatrix} = 2 \cdot (t_1 - t_0)(t_2 - t_0)(t_2 - t_1)$$

  This uses the properties of determinants:

  - swapping two rows changes a determinant by a factor of -1
  - a nonzero constant can be factored out of a single row or column
  - transpose does not change a determinant

Linearity of Determinant Operator:

The determinant of an $n \times n$ matrix produces a single number, as a function of all n^2 entries of the matrix. However, the determinant can also be considered as a function of a single row or column, with constants in all the other entries. For instance, in the 3×3 case we can write:

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} x - \begin{vmatrix} a & c \\ d & f \end{vmatrix} y + \begin{vmatrix} a & b \\ d & e \end{vmatrix} z = (bf - ce)x - (af - dc)y + (ae - bd)z,$$

which is a linear function of the first row. Further, if the first row is a sum $[x, y, z] = [x_1, y_1, z_1] + [x_2, y_2, z_2]$, then we have

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ a & b & c \\ d & e & f \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a & b & c \\ d & e & f \end{vmatrix},$$

and also:

$$\begin{vmatrix} cx & cy & cz \\ a & b & c \\ d & e & f \end{vmatrix} = c\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}.$$

Another way to write this is to let $v = [x, y, z]$, $v_1 = [x_1, y_1, z_1]$, and $v_2 = [x_2, y_2, z_2]$, and define:

$$L(v) = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}.$$

Then by the properties above, we have:

$$L(v_1 + v_2) = L(v_1) + L(v_2) \text{ and } L(cv) = cL(v),$$

and thus L is a linear operator.

Linear independence of Top-down bases

This depends on the Confluent Vandermonde determinant formula. A confluent Vandermonde determinant is a generalization of the regular Vandermonde determinant, where rows may be followed by derivatives. A few examples will give the idea:

Examples:

- A confluent Vandermonde with a regular row followed by a derivative row, followed by another regular row. The first two rows use the parameter a, and the third row uses the parameter b. For the purpose of taking derivatives, a is considered as a variable.

$$D(aab) = D(a^2b) = \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & b & b^2 \end{vmatrix} = (b-a)^2$$

- Now there are two derivative rows:

$$D(aaab) = D(a^3b) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 2 & 6a \\ 1 & b & b^2 & b^3 \end{vmatrix} = 2(b-a)^3$$

- Following the same pattern:

$$D(aaaabb) = D(a^4b^2) = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 \\ 0 & 0 & 2 & 6a & 12a^2 & 20a^3 \\ 0 & 0 & 0 & 6 & 24a & 60a^2 \\ 1 & b & b^2 & b^3 & b^4 & b^5 \\ 0 & 1 & 2b & 3b^2 & 4b^3 & 5b^4 \end{vmatrix} = 12(b-a)^8$$

In all of the above examples we can see the factor $(b-a)$ occurring to various powers. The powers are given by the product of the exponents on a and b. The constants 2 and 12 come from the diagonal entries in the a-rows.

We can also see that the case $D(abc)$ is then just the regular Vandermonde formula, which gives:

$$D(abc) = (b-a)(c-a)(c-b).$$

To adapt the regular Vandermonde formula to the confluent case, we can take all backward differences and delete those that are zero (like $(a-a)$) and then put in the diagonal factors. This yields the general formula:

$$D(a_1^{e_1}a_2^{e_2}\cdots a_n^{e_n}) = \prod_{1\le i<j\le n}(a_j-a_i)^{e_ie_j}\prod_{k=1}^{n}(e_k-1)!!$$

where the double factorial means:

$$N!! = N!(N-1)!(N-2)!\cdots 2!1!.$$

The proof of this formula is given in a separate paper on the website.