Lecture 4

Main Points:

- Properties of Bernstein Polynomials
- Cumulative Bernstein Polynomials

Properties of Bernstein Polynomials

1. Partition of Unity:
$$\sum_{i=0}^{d} B_i^d(t) = 1$$
2. Symmetry:
$$B_i^d(t) = B_{d-i}^d(1-t)$$
3. Positivity:
$$B_i^d(t) > 0, \quad \text{for } 0 < t < 1$$
4. Recursion:
$$B_i^d(t) = tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)$$
5. Derivative:
$$\frac{d}{dt}B_i^d(t) = d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right)$$

Proofs of Bernstein Properties:

1. Partition of Unity:
$$\sum_{i=0}^{d} B_i^d(t) = 1$$
This follows from the binomial theorem:
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$
with the substitution: $a = 1-t$, $b = t$, and $n = d$.
2. Symmetry:
$$B_i^d(t) = B_{d-i}^d(1-t)$$
This follows from the fact that for any binomial coefficient
$$\binom{d}{i} = \binom{d}{d-i}$$
which in turn follows from Pascal's Triangle, or also from the counting property of binomial coefficients: $\binom{d}{i}$ counts the number of subsets of size i in a set of size d, so also $\binom{d}{d-i}$ counts the number of complementary subsets of size $d-i$ in a set of size d. The subsets of size d are in $1-1$ correspondence with the complementary subsets of size $d-i$.

3. Positivity:

$$B_i^d(t) > 0, \quad \text{for } 0 < t < 1$$

   This follows directly from the definition.

4. Recursion:

$$B_i^d(t) = tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)$$

   This follows from Pascal's Identity:

$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i},$$

   and the definition of $B_i^d(t)$:

$$\begin{aligned}
B_i^d(t) &= \binom{d}{i}(1-t)^{d-i}t^i \\
&= \left(\binom{d-1}{i-1} + \binom{d-1}{i}\right)(1-t)^{d-i}t^i \\
&= \binom{d-1}{i-1}(1-t)^{d-i}t^i + \binom{d-1}{i}(1-t)^{d-i}t^i \\
&= t\binom{d-1}{i-1}(1-t)^{d-i}t^{i-1} + (1-t)\binom{d-1}{i}(1-t)^{d-i-1}t^i \\
&= tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t). \qquad (1)
\end{aligned}$$

5. Derivative:

$$\frac{d}{dt}B_i^d(t) = d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right)$$

Here we use a combination of the recursion property for Bernstein polynomials, the product rule for derivatives, and an induction argument (using the derivative formula for case $d-1$).

$$\begin{aligned}
\frac{d}{dt}B_i^d(t) &= \frac{d}{dt}\left[tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)\right] \\
&= 1 \cdot B_{i-1}^{d-1}(t) + t \cdot \frac{d}{dt}B_{i-1}^{d-1}(t) + (-1) \cdot B_i^{d-1}(t) + (1-t) \cdot \frac{d}{dt}B_i^{d-1}(t) \\
&= B_{i-1}^{d-1}(t) + t \cdot (d-1)\left(B_{i-2}^{d-2}(t) - B_{i-1}^{d-2}(t)\right) - \cdot B_i^{d-1}(t) + (1-t) \cdot (d-1)\left(B_{i-1}^{d-2}(t) - B_i^{d-2}(t)\right)) \\
&= B_{i-1}^{d-1}(t) - B_i^{d-1}(t) + (d-1)\left(tB_{i-2}^{d-2}(t) + (1-t)B_{i-1}^{d-2}(t)\right) - (d-1)\left(tB_{i-1}^{d-2}(t) + (1-t)B_i^{d-2}(t)\right)) \\
&= B_{i-1}^{d-1}(t) - B_i^{d-1}(t) + (d-1)\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right) \\
&= d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right).
\end{aligned}$$

Cumulative Bernstein polynomials and basis

Define the Cumulative Bernstein polynomials as:

$$C_i^d(t) = \sum_{j=i}^{d} B_j^d(t).$$

The Cumulative Bernstein basis is defined as:

$$\{C_0^d(t), C_1^d(t), \dots C_d^d(t)\}.$$

Note: $C_0^d(t) = 1$ by the Partition of Unity Property for Bernstein polynomials.

Examples:

- The Cumulative Bernstein basis for $d = 1$:
$$\{1, t\}.$$

- The Cumulative Bernstein basis for $d = 2$:
$$\{1, 2t(1-t) + t^2, t^2\} = \{1, 2t - t^2, t^2\}.$$

Bezier Curves

A 2D Bezier curve is simply a piecewise polynomial curve written with the polynomials in the Bernstein basis:

$$\begin{aligned}
\gamma(t) &= (p_1(t), p_2(t)) \\
&= (a_0(1-t)^2 + a_1 2(1-t)t + a_2 t^2, b_0(1-t)^2 + b_1 2(1-t)t + b_2 t^2) \\
&= (a_0, b_0)(1-t)^2 + (a_1, b_1)2(1-t)t + (a_2, b_2)t^2 \\
&= (1-t)^2(a_0, b_0) + 2(1-t)t(a_1, b_1) + t^2(a_2, b_2) \\
&= B_0^2(t)P_0 + B_1^2(t)P_1 + B_2^2(t)P_2.
\end{aligned}$$

Notes:

- The last line is the BB-form. It is a sum of points with coefficients which are Berstein polynomials evaluated at some value t. Since the Bernstein polynomials add up to 1 (the Partition of Unity Property), this is an affine sum of points.
- The third line could be called "Point-Coefficient Form". What does it mean? Can we multiply points times polynomials, as if they were scalars? The answer is no, and all we mean by this line is that it is another notation for the second line. It is simply an easy and slightly more compact way of writing the second line. It can be done for any basis of polynomials, but the Bernstein basis leads to the BB-form, which has special meaning as an affine sum.