Lecture 6

Main Points:

- Shifted (or truncated) power functions
- More bases for piecewise polynomial vector spaces

Shifted (or truncated) Power Functions:

The (right-continuous) shifted power function, with constant c, is defined as:

$$(t - c)_+^k = \begin{cases} 0, & t < c \\ (t - c)^k, & t \geq c \end{cases}, \quad \text{for } k \geq 1, \text{ and}$$

$$(t - c)_+^0 = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

The (left-continuous) shifted power function, with constant c, is defined as:

$$(c - t)_+^k = \begin{cases} 0, & c < t \\ (c - t)^k, & c \geq t \end{cases} = \begin{cases} (c - t)^k, & t \leq c \\ 0, & t > c \end{cases}, \quad \text{for } k \geq 1, \text{ and}$$

$$(c - t)_+^0 = \begin{cases} 0, & c < t \\ 1, & c \geq t \end{cases} = \begin{cases} 1, & t \leq c \\ 0, & t > c \end{cases}$$

Examples:

- The continuous function $(t - 1)_+^1$ is zero to the left of $t = 1$ and is equal to the linear function $t - 1$ to the right of $t = 1$.

- The continuous function $(1 - t)_+^1$ is zero to the right of $t = 1$ and is equal to the linear function $1 - t$ to the left of $t = 1$.

- The discontinuous (but also right-continuous) function $(t - 1)_+^0$ is zero to the left of $t = 1$ and is equal to the constant function 1 at $t = 1$ and to the right of $t = 1$.

- The discontinuous (but also left-continuous) function $(1 - t)_+^0$ is zero to the right of $t = 1$ and is equal to the constant function 1 at $t = 1$ and to the left of $t = 1$.

Bases with shifted power functions:

The standard basis of $P_d^k[u_0, \ldots, u_k]$ is:

$$\{1, t, t^2, \ldots, t^d, (t - u_1)_+^0, (t - u_1)_+^1, \ldots, (t - u_1)_+^d, \ldots, (t - u_{k-1})_+^0, (t - u_{k-1})_+^1, \ldots, (t - u_{k-1})_+^d\}.$$

Note: This basis consists of a basis of P_d (the standard basis) and also $d + 1$ shifted power functions at each of the $k - 1$ break-points.

Examples:

- A basis for $P_1^4[0, 1, 2, 3, 4]$ is:

$$\{1, t, (t - 1)_+^0, (t - 1)_+^1, (t - 2)_+^0, (t - 2)_+^1, (t - 3)_+^0, (t - 3)_+^1\}.$$

  The dimension of $P_1^4[0, 1, 2, 3, 4]$ is 8.

- A basis for $P_3^3[0,1,2,3]$ is:

$$\{1, t, t^2, t^3, (t-1)_+^0, (t-1)_+^1, (t-1)_+^2, (t-1)_+^3, (t-2)_+^0, (t-2)_+^1, (t-2)_+^2, (t-2)_+^3, \}.$$

The dimension of $P_3^3[0,1,2,3]$ is 12.

To see how these bases of shifted power functions are indeed bases, we relate them back to the the ordered k-tuples:

Correspondence between P_d^k and $P_d^k[u_0, \ldots, u_k]$:

$$(p_1, p_2, \ldots, p_k) \longleftrightarrow f(t) = \begin{cases} p_1(t), & u_0 \le t < u_1 \\ p_2(t), & u_1 \le t < u_2 \\ & \vdots \\ p_k(t), & u_{k-1} \le t < u_k \end{cases}$$

Proving that shifted power bases are indeed linearly independent:

We can use the isomorphism between P_d^k and $P_d^k[u_0, \ldots, u_k]$ and the correspondence between elements to verify that these shifted power bases are indeed linearly independent.

Examples:

- To show that $\{1, t, (t-1)_+^0, (t-1)_+^1, (t-2)_+^0, (t-2)_+^1, (t-3)_+^0, (t-3)_+^1\}$ is a basis of $P_1^4[0,1,2,3,4]$, we note that it corresponds with the basis of 4-tuples for P_1^4:

$$\{(1,1,1,1), (t,t,t,t), (0,1,1,1), (0,t-1,t-1,t-1), (0,0,1,1), (0,0,t-2,t-2), (0,0,0,1), (0,0,0,t-3)\}.$$

With the interpretation of these 4-tuples as piecewise polynomial functions on the sequence of intervals $[0,1,2,3,4]$, we see that they span the vector space $\{1, t, (t-1)_+^0, (t-1)_+^1, (t-2)_+^0, (t-2)_+^1, (t-3)_+^0, (t-3)_+^1\}$ and hence are also linearly independent.

Modified shifted power bases:

A modified basis of $P_d^k[u_0, \ldots, u_k]$ can be given with any other basis of P_d, such as $\{b_0(t), \ldots, b_d(t)\}$:

$$\{b_0(t), \ldots, b_d(t), (t-u_1)_+^0, (t-u_1)_+^1, \ldots, (t-u_1)_+^d, \ldots, (t-u_{k-1})_+^0, (t-u_{k-1})_+^1, \ldots, (t-u_{k-1})_+^d\}.$$

Examples:

- A basis of $P_2^3[1,3,5,7]$ can be given by:

$$\{1, t-1, (t-1)^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2, (t-5)_+^0, (t-5)_+^1, (t-5)_+^2\}$$

where we have simply replaced the standard basis $\{1, t, t^2\}$ with the shifted basis $\{1, t-1, (t-1)^2\}$.

- Another way to state this basis of $P_2^3[1,3,5,7]$ is also:

$$\{(t-1)_+^0, (t-1)_+^1, (t-1)_+^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2, (t-5)_+^0, (t-5)_+^1, (t-5)_+^2\}$$

where we have simply replaced the shifted basis $\{1, t-1, (t-1)^2\}$ of polynomials, by the set of shifted power functions $\{(t-1)_+^0, (t-1)_+^1, (t-1)_+^2\}$. It is important to understand that these sets of functions are identical on the interval $[1,7]$, which is the declared domain of the functions in this vector space. (A polynomial has implied domain all real numbers, but when it is a member of a vector space such as $P_2^3[1,3,5,7]$ This example illustrates that we can write bases made of functions which are all of the same type.

2

Writing a ppf in terms of a shifted power basis:

To write a given function in terms of a specific basis means to find the coordinate vector of coefficients which expresses that function as a linear combination of the basis elements. The simplest method is to start with the polynomial that defines the function on the first interval on the left, and use this to solve for some coefficients of basis functions which are nonzero on this interval. Then proceed to the next interval and work right.

Examples:

- Write the function

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 3 - t, & 2 \le t \le 3 \end{cases},$$

in terms of the basis:

$$\{1, t, (t-1)_+^1, (t-1)_+^0, (t-2)_+^1, (t-2)_+^0\}.$$

We seek coefficients $a_0, \ldots, a_5$ such that:

$$f(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Solution: We start with the interval $[0, 1)$, and with $f(t) = t$:

$$\text{interval:} \quad [0, 1), \quad f(t) = t$$

This gives us the equation:

$$t = a_0 \cdot 1 + a_1 \cdot t + \text{ zero terms.}$$

We don't need to include the other terms since they are zero on the interval $[0, 1)$. The above equation easily gives us $a_0 = 0$ and $a_1 = 1$. Our function now reads:

$$f(t) = 0 \cdot 1 + 1 \cdot t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Now we move to the next interval:

$$\text{interval:} \quad [1, 2), \quad f(t) = 1$$

This gives us the equation:

$$1 = t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + \text{ zero terms.}$$

Now when we restrict the shifted power functions to the interval $[1, 2)$ we can write them more simply as:

$$1 = t + a_2 \cdot (t-1) + a_3 \cdot 1,$$

or

$$1 - t = a_2 \cdot (t-1) + a_3 \cdot 1,$$

which has solution $a_2 = -1$ and $a_3 = 0$. Our function now reads:

$$f(t) = 0 \cdot 1 + 1 \cdot t - 1 \cdot (t-1)_+^1 + 0 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Now we move to the last interval:

$$\text{interval:} \quad [2, 3), \quad f(t) = 3 - t$$

This gives us the equation:

$$3 - t = 1 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0,$$

which simplifies to:

$$2 - t = a_4 \cdot (t-2) + a_5 \cdot 1,$$

which has solution $a_4 = -1$ and $a_5 = 0$. This gives us the complete solution:

$$f(t) = 0 \cdot 1 + 1 \cdot t - 1 \cdot (t-1)_+^1 + 0 \cdot (t-1)_+^0 - 1 \cdot (t-2)_+^1 + 0 \cdot (t-2)_+^0.$$

So the coordinate vector of f with respect to this basis is:

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$