Lecture 8

Main Points:

- Recursive form and third proof of existence and uniqueness
- Divided differences
- Newton form and basis
- Leibniz formula

Advantages of the Newton Form

The two previous forms of the interpolating polynomial, in standard basis and Lagrange basis, have some disadvantages which can be removed by considering the Newton form. In particular, adding a new interpolation point in the previous two cases requires setting up a new linear system or a new set of Lagrange polynomials. In the case of the Newton form, we can recycle all of our work and simply add one more term to the interpolating polynomial in order to match one more point. Before we define the Newton form, we will first give another method to prove the existence and uniqueness of the interpolating polynomial, which will then give rise to the Newton form.

Third proof of Existence and Uniqueness of Interpolating Polynomial.

This proof is based on a recursive formula and an induction argument. So we will need a base case, with $d = 0$. The polynomial interpolation problem for one data point is trivial. In particular, to find the polynomial of degree zero that matches a data function $g(t)$ at t_0, we simply write the constant function

$$p(t) = g(t_0).$$

This is the base case. Now, for the induction step we make the assumption that the interpolating polynomial exists for degrees $\leq d$. This is the induction hypothesis. Then assuming that we have data values $t_0, \ldots, t_d$, and data function $g(t)$, we can define, based on the induction hypothesis, two polynomials of degree $\leq d-1$:

$$p_0(t), \quad \text{with data values } \; t_0, t_1, \ldots, t_{d-1},$$

and

$$p_1(t), \quad \text{with data values } \; t_1, t_2, \ldots, t_d.$$

This means that

$$p_0(t_i) = g(t_i), \; i = 0, \ldots, d-1, \quad \text{and} \quad p_1(t_i) = g(t_i), \; i = 1, \ldots, d.$$

Assuming the existence of these two polynomials, we then define:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

One easily checks now that

$$p(t_0) = p_0(t_0) = g(t_0) \quad \text{and} \quad p(t_d) = p_1(t_d) = g(t_d).$$

Then we can also check the middle values t_i with $1 \leq i \leq d-1$, using the fact that for such i we have $p_0(t_i) = g(t_i) = p_1(t_i)$:

$$\begin{aligned} p(t_i) &= \frac{t_i - t_0}{t_d - t_0} p_1(t_i) + \frac{t_d - t_i}{t_d - t_0} p_0(t_i) \\ &= \frac{t_i - t_0}{t_d - t_0} g(t_i) + \frac{t_d - t_i}{t_d - t_0} g(t_i) \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{t_i - t_0}{t_d - t_0} + \frac{t_d - t_i}{t_d - t_0}\right] g(t_i) \\
&= \left[\frac{t_d - t_0}{t_d - t_0}\right] g(t_i) \\
&= g(t_i)
\end{aligned}$$

Note that since $p_0(t)$ and $p_1(t)$ each have degree $\leq d-1$ it must be that $p(t)$ is a sum of polynomials of degree $\leq d$ and hence $p(t)$ is in P_d. This shows that the interpolating polynomial $p(t)$ exists in P_d.

To prove again the uniqueness of the interpolating polynomial, we suppose that two such polynomials $p(t)$ and $q(t)$ exist in P_d, and define:

$$f(t) = p(t) - q(t).$$

Then $f(t)$ is also in P_d and since $p(t)$ and $q(t)$ satisfy the interpolation conditions, we have:

$$f(t_i) = p(t_i) - q(t_i) = g(t_i) - g(t_i) = 0, \quad i = 0, \ldots, d.$$

Thus $f(t)$ is a polynomial in P_d with $d+1$ distinct zeros $t_0, \ldots, t_d$. But a nonzero polynomial of degree at most d can have at most d zeros, since each zero corresponds to a factor of f. Only the zero polynomial in P_d can have more than d zeros, so in fact it must be that $f(t) = 0(t)$ which means that $p(t) = q(t)$. This shows that the interpolating polynomial is unique in P_d.

Definition of the Newton basis:

Given d real numbers $t_0, \ldots, t_{d-1}$, which may or may not be distinct, we define the Newton basis of P_d as

$$\{1, t - t_0, (t - t_0)(t - t_1), (t - t_0)(t - t_1)(t - t_2), \ldots, (t - t_0)(t - t_1)(t - t_2)\cdots(t - t_{d-1})\}.$$

This can easily be seen to be linearly independent since the elements are of increasing degree, giving a triangular matrix of coordinate vectors with respect to the standard basis, with determinant 1.

Examples:

- The Newton basis of P_2 for the values $t_0 = 1$ and $t_1 = 3$ is:
$$\{1, t - 1, (t - 1)(t - 3)\}.$$
- The Newton basis of P_2 for the values $t_0 = 3$ and $t_1 = 3$ is:
$$\{1, t - 3, (t - 3)^2\},$$
which is also a shifted basis.

In order to define the Newton form we need to define divided differences. The divided difference is a number which is obtained from the data specified for a polynomial interpolation problem. In other words, the input is the data $t_0, t_1, \ldots, t_d$ and g, and the output is the number $[t_0, t_1, \ldots, t_d]g$. As we see below, this number is defined using the interpolating polynomial $p(t)$.

Definition of divided differences:

The divided difference $[t_0, t_1, \ldots, t_d]g$ is defined to be the coefficient of t^d in the interpolating polynomial $p(t)$ in P_d with data function g and data values $t_0, t_1, \cdots, t_d$. In other words, if $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$ in standard basis form, then

$$[t_0, t_1, \ldots, t_d]g = a_d.$$

Note: because the interpolating polynomial depends only on the input data and data function $g(t)$, it is not dependent on the order of the data. This also means that the order of the data values in the divided difference can be changed without affecting the outcome.

Examples:

- Find the divided difference $[0,1,2]g$ with $g(t) = 2^t$. Since we already worked out the interpolating polynomial $p(t)$ above to be $1 + \frac{1}{2}t + \frac{1}{2}t^2$, we see from the definition that $[0,1,2]$ is the coefficient of t^2 which is $\frac{1}{2}$, so

  $$[0,1,2]g = \frac{1}{2}.$$

  Note: From the comments above, since we are only taking the coefficient of t^2, the input data can be in any order, so we have:

  $$[0,1,2]g = [0,2,1]g = [1,0,2]g = [1,2,0]g = [2,0,1]g = [2,1,0]g = \frac{1}{2}.$$

- Find the divided difference $[0,1]$ with $g(t) = 2^t$. For this data, the interpolating polynomial is just the line through the points $(0, g(0))$ and $(1, g(1))$, or $(0,1)$ and $(1,2)$. This line is $p(t) = 1 + t$. So

  $$[0,1]g = 1.$$

Since the sequence $t_0, t_1, \ldots, t_d$ has many possible subsequences, each of which may be used in the recursion below, we may write $t_i, t_{i+1}, \ldots, t_{i+k}$ to represent the most general such sequence.

Recursion Property for divided differences:

The divided differences, defined above, also satisfy the recursion:

$$[t_0, t_1, \ldots, t_d]g = \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0}.$$

Similarly:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]g = \frac{[t_{i+1}, t_{i+2}, \ldots, t_{i+k}]g - [t_i, t_{i+1}, \ldots, t_{i+k-1}]g}{t_{i+k} - t_i}.$$

Examples:

- The divided differences can be computed in the following table format, here for $d = 2$.

  $$\begin{array}{llll}
t_0 & [t_0]g = g(t_0) & & \\
 & & [t_0,t_1]g = \frac{[t_1]g-[t_0]g}{t_1-t_0} & \\
t_1 & [t_1]g = g(t_1) & & [t_0,t_1,t_2]g = \frac{[t_1,t_2]g-[t_0,t_1]g}{t_2-t_0} \\
 & & [t_1,t_2]g = \frac{[t_2]g-[t_1]g}{t_2-t_1} & \\
t_2 & [t_2]g = g(t_2) & &
\end{array}$$

- Here is an example with specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 3$, $g(1) = -2$, and $g(2) = 1$.

  $$\begin{array}{llll}
0 & 3 & & \\
 & & \frac{-2-3}{1-0} = -5 & \\
1 & -2 & & \frac{3-(-5)}{2-0} = 4 \\
 & & \frac{1-(-2)}{2-1} = 3 & \\
2 & 1 & &
\end{array}$$

Proof of the Recursive Property

To prove the recursive property for divided differences we use the recursive form of the interpolating polynomial:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t),$$

where the polynomials $p_0(t)$ and $p_1(t)$ in P_{d-1} are also interpolating polynomials:

$$p_0(t), \quad \text{with data values } \ t_0, t_1, \ldots, t_{d-1},$$

and

$$p_1(t), \quad \text{with data values } \ t_1, t_2, \ldots, t_d.$$

Now we recall the definition of the operator $[t_0, \ldots, t_d]g$ as the coefficient of t^d in $p(t)$. To extract the coefficients of t^d we suppose that

$$p_0(t) = a_0 + a_1 t + \cdots + a_{d-1} t^{d-1},$$

and

$$p_1(t) = b_0 + b_1 t + \cdots + b_{d-1} t^{d-1}.$$

Then:

$$p(t) = \frac{t - t_0}{t_d - t_0} \left(a_0 + a_1 t + \cdots + a_{d-1} t^{d-1} \right) + \frac{t_d - t}{t_d - t_0} \left(b_0 + b_1 t + \cdots + b_{d-1} t^{d-1} \right).$$

By the induction hypothesis, we also can say that

$$a_{d-1} = [t_0, t_1, \ldots, t_{d-1}]g,$$

and

$$b_{d-1} = [t_1, t_2, \ldots, t_d]g.$$

So, the coefficient of t^d in $p(t)$ is

$$\begin{aligned} [t_0, t_1, \ldots, t_d]g &= \frac{1}{t_d - t_0} a_{d-1} + \frac{-1}{t_d - t_0} b_{d-1} \\ &= \frac{1}{t_d - t_0} [t_0, t_1, \ldots, t_{d-1}]g + \frac{-1}{t_d - t_0} [t_1, t_2, \ldots, t_d]g \\ &= \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0}. \end{aligned}$$

Newton Form of the interpolating polynomial

The expansion of the interpolating polynomial $p(t)$ with data $t_0, t_1, \cdots, t_d$ and g, in terms of the Newton basis is called the Newton Form, and can be written as:

$$p(t) = \sum_{i=0}^{d} [t_0, \ldots, t_i] g N_i(t),$$

where $N_0(t) = 1$, $N_1(t) = t - t_0$, $N_2(t) = (t - t_0)(t - t_1)$, $\ldots$, and $N_d(t) = (t - t_0)(t - t_1) \cdots (t - t_{d-1})$.

Examples:

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 2$, $t_1 = 4$, and $g(2) = 6$, $g(4) = -2$. The divided difference table is:

$$\begin{array}{lll} t_0 & [t_0]g & \\ & & [t_0, t_1]g \\ t_1 & [t_1]g & \end{array} \qquad \text{or} \qquad \begin{array}{lll} 2 & 6 & \\ & & \frac{-2-6}{4-2} = -4 \\ 4 & -2 & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0, t_1]g \cdot (t - t_0) \\ &= 6 + (-4) \cdot (t - 2) \\ &= 14 - 4t, \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(2) = 6$, $p(4) = -2$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 3$, $g(1) = -2$, and $g(2) = 1$. The divided difference table which we computed in a previous example is:

$$\begin{array}{ccccc} t_0 & [t_0]g & & & \\ & & [t_0,t_1]g & & \\ t_1 & [t_1]g & & [t_0,t_1,t_2]g & \\ & & [t_1,t_2]g & & \\ t_2 & [t_2]g & & & \end{array} \quad \text{or} \quad \begin{array}{ccccc} 0 & 3 & & & \\ & & \frac{-2-3}{1-0} = -5 & & \\ 1 & -2 & & \frac{3-(-5)}{2-0} = 4 & \\ & & \frac{1-(-2)}{2-1} = 3 & & \\ 2 & 1 & & & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) \\ &= 3 + (-5) \cdot (t - 0) + 4 \cdot (t - 0)(t - 1) \\ &= 3 - 5t + 4t(t - 1), \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0) = 3$, $p(1) = -2$, and $p(2) = 1$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, and $g(0) = 2$, $g(1) = 3$, $g(2) = 4$, and $g(3) = -1$. The divided difference table is:

$$\begin{array}{ccccccc} t_0 & [t_0]g & & & & & \\ & & [t_0,t_1]g & & & & \\ t_1 & [t_1]g & & [t_0,t_1,t_2]g & & & \\ & & [t_1,t_2]g & & [t_0,t_1,t_2,t_3]g & & \\ t_2 & [t_2]g & & [t_1,t_2,t_3]g & & & \\ & & [t_2,t_3]g & & & & \\ t_3 & [t_3]g & & & & & \end{array} \quad \text{or} \quad \begin{array}{ccccc} 0 & 2 & & & \\ & & \frac{3-2}{1-0} = 1 & & \\ 1 & 3 & & \frac{1-1}{2-0} = 0 & \\ & & \frac{4-3}{2-1} = 1 & & \frac{-3-0}{3-0} = -1 \\ 2 & 4 & & \frac{-5-1}{3-1} = -3 & \\ & & \frac{-1-4}{3-2} = -5 & & \\ 3 & -1 & & & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1)[t_0, t_1, t_2, t_3]g \cdot (t - t_0)(t - t_1)(t - t_2) \\ &= 2 + 1 \cdot (t - 0) + 0 \cdot (t - 0)(t - 1) - 1 \cdot (t - 0)(t - 1)(t - 2) \\ &= 2 + t - t(t - 1)(t - 2), \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0) = 2$, $p(1) = 3$, $p(2) = 4$ and $p(3) = -1$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 2$, $g(1) = 3$, and $g(2) = 4$. Note: This is just the first three data points from the previous example. Also note: the three points are collinear, so we already know that there is a linear polynomial passing through them. The divided difference table is:

$$\begin{array}{cccc} 0 & 2 & & \\ & & \frac{3-2}{1-0}=1 & \\ 1 & 3 & & \frac{1-1}{2-0}=0 \\ & & \frac{4-3}{2-1}=1 & \\ 2 & 4 & & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0,t_1]g\cdot(t-t_0) + [t_0,t_1,t_2]g\cdot(t-t_0)(t-t_1) \\ &= 2 + 1\cdot(t-0) + 0\cdot(t-0)(t-1) \\ &= 2+t, \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0)=2$, $p(1)=3$, $p(2)=4$.

Leibniz' Rule for Divided Differences

Let $f(t)=g(t)h(t)$. Then

$$[t_i,t_{i+1},\ldots,t_{i+k}]f = \sum_{r=i}^{i+k}([t_i,\ldots,t_r]g)([t_r,\ldots,t_{i+k}]h).$$

Note: Draw the divided difference triangles for g and h, and label them T_1 and T_2. Then make two lists, first along the top of T_1 from left to right, then along the bottom of T_2, from right to left. Each of these lists has length $k+1$ and can be viewed as a vector. The dot product of these two vectors is the same as the sum above.

Examples:

- For degree $d=2$ the Leibniz formula looks like this:

$$[t_0,t_1,t_2]f = [t_0]g[t_0,t_1,t_2]h + [t_0,t_1]g[t_1,t_2]h + [t_0,t_1,t_2]g[t_2]h.$$

  The coefficients come from the two divided difference tables:

$$\begin{array}{cccc} t_0 & [t_0]g & & \\ & & [t_0,t_1]g & \\ t_1 & [t_1]g & & [t_0,t_1,t_2]g \\ & & [t_1,t_2]g & \\ t_2 & [t_2]g & & \end{array} \quad \text{and} \quad \begin{array}{cccc} t_0 & [t_0]h & & \\ & & [t_0,t_1]h & \\ t_1 & [t_1]h & & [t_0,t_1,t_2]h \\ & & [t_1,t_2]h & \\ t_2 & [t_2]h & & \end{array}$$

- Let $f(t)=|t|(t-2)^2$, with $g(t)=|t|$, and $h(t)=(t-2)^2$, and take $t_0=-1$, $t_1=0$, and $t_2=1$. We then have the tables:

$$\begin{array}{cccc} -1 & 1 & & \\ & & -1 & \\ 0 & 0 & & 1 \\ & & 1 & \\ 1 & 1 & & \end{array} \quad \text{and} \quad \begin{array}{cccc} -1 & 9 & & \\ & & -5 & \\ 0 & 4 & & 1 \\ & & -3 & \\ 1 & 1 & & \end{array}$$

  Then we have the corresponding Leibniz formula for $[-1,0,1]f$:

$$\begin{aligned}
[-1,0,1]f &= [-1]g[-1,0,1]h + [-1,0]g[0,1]h + [-1,0,1]g[1]h \\
&= (1)(1) + (-1)(-3) + (1)(1) \\
&= 5
\end{aligned}$$

We can confirm this by constructing the table for f alone:

$$\begin{array}{ccccc}
-1 & 9 & & & \\
 & & -9 & & \\
0 & 0 & & 5 & \\
 & & 1 & & \\
1 & 1 & & &
\end{array}$$

This confirms directly that $[-1,0,1]f = 5$.

- Let $f(t) = (t-2)_+^2(t-2)$, with $g(t) = (t-2)_+^2$, and $h(t) = t-2$, and take $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$. We then have the tables:

$$\begin{array}{cccc}
1 & 0 & & \\
 & & 0 & \\
2 & 0 & & \frac{1}{2} \\
 & & 1 & \\
3 & 1 & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{cccc}
1 & -1 & & \\
 & & 1 & \\
2 & 0 & & 0 \\
 & & 1 & \\
3 & 1 & &
\end{array}$$

Then we have the corresponding Leibniz formula for $[1,2,3]f$:

$$\begin{aligned}
[1,2,3]f &= [1]g[1,2,3]h + [1,2]g[2,3]h + [1,2,3]g[3]h \\
&= (0)(0) + (0)(1) + (\frac{1}{2})(1) \\
&= \frac{1}{2}
\end{aligned}$$

We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc}
1 & 0 & & \\
 & & 0 & \\
2 & 0 & & \frac{1}{2} \\
 & & 1 & \\
3 & 1 & &
\end{array}$$