Multidimensional action of cellular automata on music sequences

An implemantation in UPISketch

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Abstract. A natural question in music history is how to get a composition extracted by one main element often called the thema like in a fuga or a scale. The idea is to exploit at the maximum of its possibilities this object to generate new elements related to it by analysing all its dimensions like rythms, intervals, amplitude, timbre. A simple exemple is the way to build counter subjects in the fuga. When we build such new elements, one really natural way to do computations on the original thema is to compute things locally, notes by notes, doing a self study. Cellular automata are transformations that act locally and we will investigate the case of two automata in particular, one computing symetries and the other one, the intervals. We will present some results, implementations and ideas of which automata can be used in a natural way in music.

1 Background and objects

A dynamical system is a triplet $(\mathcal{X}, \mathcal{T}, F)$ with $\mathcal{X}$ the **space of phases** (or configurations), whose elements or **points** represent the possible states of the system. $\mathcal{T}$ representing the **time**, which can be discrete or continuous. The latter can extend into the future (irreversible processes) or into the past and the future (reversible processes). $F : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ is the law of evolution and let introduce the set $\mathcal{O}_{\boldsymbol{x}} = \{F(\boldsymbol{x}, t) : t \in \mathcal{T}\}$, the **orbit** starting from the point $\boldsymbol{x}$. We note $F(\boldsymbol{x}, t) := F^t(\boldsymbol{x})$. One crucial question in this field is to be able to describe the orbit $\mathcal{O}_{\boldsymbol{x}}$, is it a finite set ? Does it contains cycles ? Is it chaotic ? What is the importance of the starting point ? Cellular automata belong to the big family of dynamical systems.

1.1 Cellular automata

A 1-D **deterministic cellular automaton** is described by a neighbourhood $\mathcal{N} \subset \mathbb{Z}$ and a set $\mathbb{A}$ called alphabet with $\mathcal{X} = \mathbb{A}^{\mathbb{Z}}$, it is a transformation $F : \mathbb{A}^{\mathbb{Z}} \to \mathbb{A}^{\mathbb{Z}}$ defined by its **local function** $f : \mathbb{A}^{\mathcal{N}} \to \mathbb{A}$ by the relation :

$$F(\boldsymbol{x})_i = f((\boldsymbol{x}_{i+k})_{k \in \mathcal{N}})$$

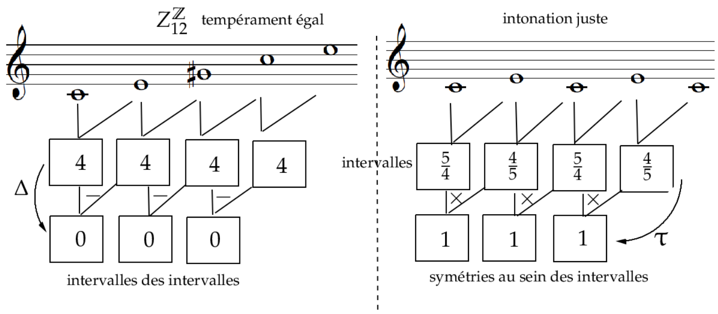
For 2-D automata, $\mathbb{Z}$ must be replaced by $\mathbb{Z}^2$. A simple but essential CA is called the **shift map** acting the following way :

$$\forall \boldsymbol{x} \in \mathbb{A}^{\mathbb{Z}} : \sigma(\boldsymbol{x})_i = \boldsymbol{x}_{i+1}.$$

Let introduce two automata with $\mathcal{N} = \{0,1\}$, $\tau = \sigma + \mathrm{id}$ and $\Delta = \sigma - \mathrm{id}$. The first one is called the *Ledrappier automaton* and the second one the *Vieru automaton*. $f_\tau(a,b) = a + b$ and $f_\Delta(a,b) = b - a$. Δ is acting on a musical sequence by **comparing succesively** each of his terms when τ on the other hand is acting by **computing the symetry** between each consecutive terms. For all $x \in \mathcal{X}$:

$$\Delta(x)_i = x_{i+1} - x_i \text{ and } \tau(x)_i = x_i + x_{i+1}$$

$$\Delta(x) = \sigma(x) - x \text{ and } \tau(x) = x + \sigma(x)$$



In the case of a finite sequence $x \in \mathbb{A}^\ell$ for $\ell \in \mathbb{N}^*$, note that $F(x)$ is of size $\ell - 1$. A 5 notes sequences gives 4 intervals. It is important to understand the $+$ and the $-$ operations can be more complex than computing intervals (like in [4]) as $\mathbb{A}$ can be multi dimensional as explained in 3. We could furthemore note the law of association $*$ and note that

$$\Delta(x) = x^- * \sigma(x) \text{ and } \tau(x) = x * \sigma(x)$$

Remark 1. A natural transformation which is not a CA but related to Δ is $\tilde{\Delta}(x, y) = y - x$. $\Delta(x) = \tilde{\Delta}(x, \sigma(x))$.

Those automata are called **sigma-polynomial automata** which means there are polynomial of σ. It is thus easy to compute their images and preimages like in [2,1] which are given by :

$$\forall n > 0, \tau^n(x) = \sum_{k=0}^{n} \binom{n}{k} \sigma^k(x) \text{ and } \Delta^n(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sigma^k(x)$$
$$\forall i \in \mathbb{Z}, \tau^n(x)_i = \sum_{k=0}^{n} \binom{n}{k} x_{i+k} \text{ and } \Delta^n(x)_i = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} x_{i+k}$$

Let explicit the construction of the preimages for $F \in \{\Delta, \tau\}$: let introduce for a given x, the set $F^{-1}(x) = \{y \in \mathbb{A}^{\mathbb{Z}} | F(y) = x\}$ and for a given $a \in \mathbb{A}$ the application

$$F_a^{-1} : \begin{array}{l} \mathbb{A}^{\mathbb{Z}} \to \mathbb{A}^{\mathbb{Z}} \\ x \mapsto y \in F^{-1}(x) \text{ and } y_0 = a. \end{array}$$

We define the preimages iteration recursively with the sequence of elements $A_n = (a_{-1}, a_{-2}, ..., a_n)$ that can be deterministic or probabilistic by

$$F^n_{A_n}(\boldsymbol{x}) = F^{-1}_{a_n} \circ F^{-1}_{a_{n+1}} \circ ... \circ F^{-1}_{a_{-1}}(\boldsymbol{x}).$$

It has been showed in [1] that as Δ and $\boldsymbol{\tau}$ are **bipermutative** CA that F_a^{-1} is a bijection. For $n \leq 0$ we have that

$$\begin{aligned} \Delta^n_{A_n}(\boldsymbol{x})_i &= \sum_{j=0}^{|n|-1} \binom{i}{j} a_{n+j} + \sum_{k=0}^{n+i} \binom{i-k-1}{|n|-1} \boldsymbol{x}_k \\ \text{and } \boldsymbol{\tau}^n_{A_n}(\boldsymbol{x})_i &= \sum_{j=0}^{|n|-1} (-1)^{i+j} \binom{i}{j} a_{n+j} + \sum_{k=0}^{n+i} (-1)^{i+n+k} \binom{i-k-1}{|n|-1} \boldsymbol{x}_k. \end{aligned}$$

We consider also their noisy versions called Δ_ε and $\boldsymbol{\tau}_\varepsilon$ which are **probabilistic cellular automata**. With probability $\varepsilon \in \mathbb{R}$ a small value, the automata will deviate from its deterministic result commiting an error.

Let $\mathcal{U}_\mathbb{A}$ be the uniform distribution on $\mathbb{A}$. We give an exemple of how $\boldsymbol{\tau}_\varepsilon$ and Δ_ε can be build :

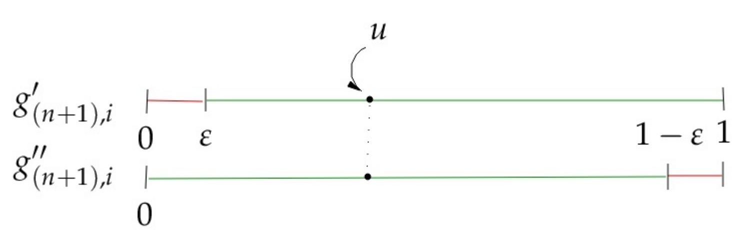
$$\begin{aligned} f_{\tau_\varepsilon} : \mathbb{A}^2 &\to \mathcal{M}(\mathbb{A}) & f_{\Delta_\varepsilon} : \mathbb{A}^2 &\to \mathcal{M}(\mathbb{A}) \\ (a,b) &\mapsto (1-\varepsilon)\delta_{\{a+b\}} + \varepsilon \mathcal{U}_{\mathbb{A}\setminus\{a+b\}} & (a,b) &\mapsto (1-\varepsilon)\delta_{\{b-a\}} + \varepsilon \mathcal{U}_{\mathbb{A}\setminus\{b-a\}} \end{aligned}$$

Let $X = (X_{n,i})_{n,i \in \mathbb{Z}}$ be a stochastic processes following the dynamic of $\boldsymbol{\tau}_\varepsilon$ and $Y = (Y_{n,i})_{n,i \in \mathbb{Z}}$ one following the dynamic of Δ_ε. We can define those processes as Markov ones, which recursively gives for all time $n \in \mathbb{Z}$:

$$X_{n+1} := \boldsymbol{\tau}(X_n) + \xi_{n+1} \quad \text{and} \quad Y_{n+1} := \Delta(Y_n) + \xi_{n+1}$$

where $(\xi_n(i))_{i,n \in \mathbb{Z}}$ is the error process. In our exemple we can choose for all $n, i \in \mathbb{Z}$ that $\mathcal{L}(\xi_{n,i}) = (1-\varepsilon)\delta_{0_\mathbb{A}} + \varepsilon\, \mathcal{U}_{\mathbb{A}\setminus\{0_\mathbb{A}\}}$. An other natural choice for the error process can be to choose a normal law, centered in $0_\mathbb{A}$ meaning smaller mistakes are more frequent than big ones.

We define error by $\xi_n(i) := g_{n,i}(U_n(i))$ with $\mathcal{L}(U_n(i)) = \mathcal{U}[0,1]$. The choice on the $g_{n,i}$ can vary according to the site (n,i), but also depend on how the process behaved until time $n-1$.



2 Results, uses & exemples

We describe now some exemples of action of Δ and τ on sequences which are natural to explore and some techniques coming from the CA theory that are interesting to explore musicaly. We proceed in a gradual approach in regard to the complexity.

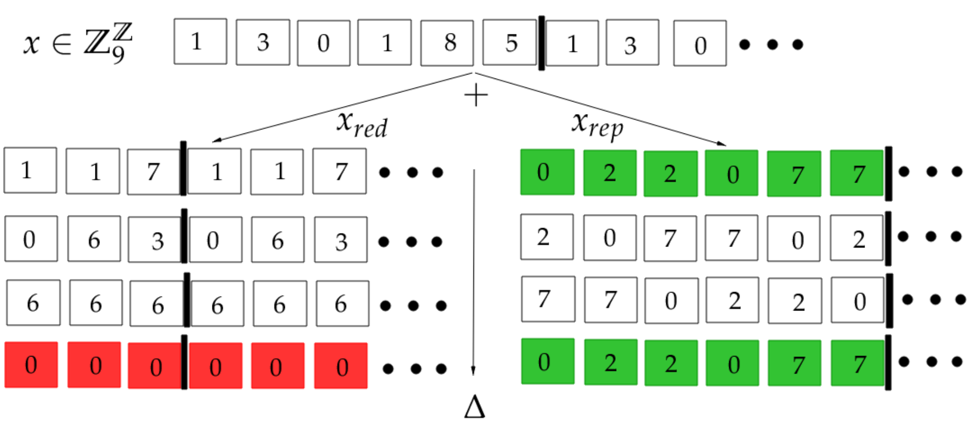
2.1 Periodic sequences

Let $\mathbb{A} = \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ for the periodic case. let $\ell \geq 1$, a music sequence $\boldsymbol{x}$ is said to be ℓ-**periodic** if for all $i \in \mathbb{Z}$, $\boldsymbol{x}_i = \boldsymbol{x}_{i+\ell}$, i.e $\sigma^\ell(\boldsymbol{x}) = \boldsymbol{x}$. We defined now the function $\pi(\boldsymbol{x}) := \min\{\ell \geq 1 / \sigma^\ell \boldsymbol{x} = \boldsymbol{x}\}$.

If $\boldsymbol{x}$ is ℓ-periodic then $\pi(\boldsymbol{x})|\ell$: ℓ is not necessarily the minimum period. We can therefore say that $\mathcal{C}_N^\ell = \{\boldsymbol{x} \in \mathbb{Z}_N^{\mathbb{Z}} / \pi(\boldsymbol{x})|\ell\}$. When we designate the minimal period $\pi(\boldsymbol{x})$, we will say that **the period of $\boldsymbol{x}$ is** ℓ or that $\boldsymbol{x}$ **is of period** ℓ. We define the set of sequences of period ℓ by $\mathcal{P}_N^\ell := \{\boldsymbol{x} \in \mathbb{Z}_N^{\mathbb{Z}} | \pi(\boldsymbol{x}) = \ell\}$. Of course $\mathcal{P}_N^\ell \subset \mathcal{C}_N^\ell$. Two fundamental sets are $\mathrm{Rep}_F := \bigcup_{n \in \mathbb{N}} \ker(F^n - \mathrm{id})$ and $\mathrm{Red}_F := \bigcup_{n \in \mathbb{N}} \ker(F^n)$. Let $\mathrm{Red}_F(\mathcal{C}_N^\ell) := \mathcal{C}_N^\ell \bigcap \mathrm{Red}_F$ and $\mathrm{Rep}_F(\mathcal{C}_N^\ell) := \mathcal{C}_N^\ell \bigcap \mathrm{Rep}_F$. Let $t_{\mathrm{red}} := \min\{t \geq 0 / F^t(\boldsymbol{x}) = \boldsymbol{0}\}$ and $t_{\mathrm{rep}} := \min\{t \geq 0 / F^t(\boldsymbol{x}) = \boldsymbol{x}\}$.

Theorem 1. *Let $F \in \{\Delta, \tau\}$ and $\ell \in \mathbb{N}^*$. Every sequence $\boldsymbol{x} \in \mathcal{C}_N^\ell$ can be decomposed in a unique way into $\boldsymbol{x} = \boldsymbol{x}_{red} + \boldsymbol{x}_{rep}$, with $\boldsymbol{x}_{red} \in \mathrm{Red}_F(\mathcal{C}_N^\ell)$ and $\boldsymbol{x}_{rep} \in \mathrm{Rep}_F(\mathcal{C}_N^\ell)$.*

$$\mathcal{C}_N^\ell = \mathrm{Rep}_F(\mathcal{C}_N^\ell) \oplus \mathrm{Red}_F(\mathcal{C}_N^\ell).$$



Some results were found in [3] and then improved in [1].

We present now a way to translate all the results found on Δ to τ : let ζ be the endomorphism of $\mathbb{Z}_N^{\mathbb{Z}}$, that we call the **alternate**, defined by $\zeta : \boldsymbol{x} \mapsto ((-1)^i \boldsymbol{x}_i)_{i \in \mathbb{Z}}$. We can remark that $\zeta^{-1} = \zeta$ because $\zeta^2 = \mathrm{id}$. we give a relation allowing to link first of all $-\Delta$ et $\boldsymbol{\tau}$ grâce à ζ: $\boldsymbol{\tau} = \zeta \circ (-\Delta) \circ \zeta$ and we have the following conjugation relation:

$$(-\Delta) \circ \zeta = \zeta \circ \boldsymbol{\tau}.$$

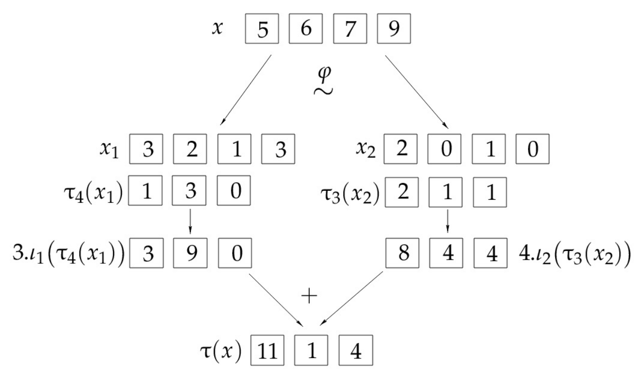
Let $N = p_1^{k_1} ... p_\ell^{k_\ell}$ where $(p_i)_{i \in \{1,...,\ell\}}$ are the prime factors of N. We can split $\mathbb{Z}_N$ thanks to its subgroups. Indeed, let take $\mathbb{Z}_{12}$ for instance:

$$\mathbb{Z}_{12} = \frac{12}{2^2}\mathbb{Z}_{12} \oplus \frac{12}{3}\mathbb{Z}_{12} = 3\mathbb{Z}_{12} \oplus 4\mathbb{Z}_{12} = \{0,3,6,9\} \oplus \{0,4,8\} := G_4 \oplus G_3.$$

$$\mathbb{Z}_N^{\mathbb{Z}} = \bigoplus_{i \in \{1,...,\ell\}} G_{p_i^{k_i}}^{\mathbb{Z}} \overset{\varphi}{\sim} \bigotimes_{i \in \{1,...,r\}} \mathbb{Z}_{p_i^{k_i}}^{\mathbb{Z}}.$$

Let $\boldsymbol{x} \in \mathbb{A}^{\mathbb{Z}}$ and note $\varphi(\boldsymbol{x}) = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_r)$ its decomposition in $\bigotimes_{i \in \{1,...,r\}} \mathbb{Z}_{p_i^{k_i}}^{\mathbb{Z}}$.

– The sequence $\boldsymbol{x}$ is reducible for F if and only if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_r$ are for $F_1, \dots, F_r$ respectively. In this case, $t_{\text{red}}(F) = \max_{i=1}^r t_{\text{red}}(F_i)$.
– The sequence $\boldsymbol{x}$ is reproducible for F if and only if if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_r$ are for $F_1, \dots, F_r$ respectively. In this case, $t_{\text{rep}}(F) = \bigvee_{i=1}^r t_{\text{red}}(F_i)$.



Theorem 2 ([1]).
Let $\boldsymbol{x} \in \mathbb{Z}_{p^k}^{\mathbb{Z}}$ is :
– *Δ-reductible iff $\pi(\boldsymbol{x}) = p^m$ and $t_{\text{red}}(\boldsymbol{x}) \leq k\pi(\boldsymbol{x})$.*
– *if p prime, odd, $k \geq 1$ then $\boldsymbol{x}$ $\boldsymbol{\tau}$-reductible iff $\overline{\pi}(\boldsymbol{x}) = p^m$ and $t_{\text{red}}(\boldsymbol{x}) \leq k\overline{\pi}(\boldsymbol{x})$.*
– *if $p = 2$, $k \geq 1$ then $\boldsymbol{x}$ $\boldsymbol{\tau}$-réductible iff $\pi(\boldsymbol{x}) = 2^m$ and $t_{\text{red}}^{\tau}(\boldsymbol{x}) \leq k\pi(\boldsymbol{x})$.*

Let $P_{d,\ell}^{\Delta}(\boldsymbol{x}) := \sum_{i=0}^{\frac{\ell}{d}-1} \sigma^{di}\boldsymbol{x}$ and $P_{d,\ell}^{\tau}(\boldsymbol{x}) := P_{d,\ell}^{\Delta}(\zeta(\boldsymbol{x})) = \sum_{i=0}^{\frac{\ell}{d}-1} (-1)^{di}\sigma^{di}\boldsymbol{x}$.

– *$\boldsymbol{x}$ est Δ-reproductible $\Longleftrightarrow$ $P_{p^m,\ell}^{\Delta}(\boldsymbol{x}) = \boldsymbol{0}$ with $m \geq 0$ the greatest integer such that $p^m | \ell$.*
– *$\boldsymbol{x}$ est $\boldsymbol{\tau}$-reproductible $\Longleftrightarrow$ $\zeta(\boldsymbol{x})$ est Δ-reproductible .*
– *If p and ℓ are prime together with $p \neq \ell$. We note $\ell = p^m q$. Let $\boldsymbol{x} \in \mathbb{Z}_p^{\mathbb{Z}}$, Δ-reproductible and of period ℓ. We note $O_q(p)$ the order of p in the multiplicative group $\left((\mathbb{Z}_q)^{\times}, \times\right)$. Then*

$$t_{rep}^{\Delta}(\boldsymbol{x}) | p^m (p^{O_q(p)} - 1).$$

We can describe the orbit $\mathcal{O}_x$ and thus respond to the question raised in 1 .

2.2 Sturmian sequences

Sturmian sequences have been widely used in music [5,**?**] because they are the less complex sequences which are not aperiodic. They are usefull to generate rythms with 0 meaning a silence and 1 the presence of a note.

The **complexity function** of an infinite sequence $\boldsymbol{x} \in \mathbb{A}^{\mathbb{Z}}$ $\mathbb{A}$ is the function C that counts the number of factors (words) of lenght ℓ in $\boldsymbol{x}$: $\mathrm{C}(\boldsymbol{x}, \ell) = Card(\mathbf{W}_\ell(\boldsymbol{x}))$, where $\mathbf{W}_\ell(\boldsymbol{x})$ is the set of word of size ℓ present in $\boldsymbol{x}$.

A sequence $\boldsymbol{x}$ is sturmian iff for all $\ell \geq 0 : \mathrm{C}(\boldsymbol{x}, \ell) = \ell + 1$. We give the exemple of the famous Fibonnaci sequence :

$$\boldsymbol{t}_{\mathrm{fib}} = 0100101001001010010100100101001001\ldots$$

The frequency of repetition of a word w is noted $r_w(\boldsymbol{x})$ and defined $r_w(\boldsymbol{x}) = \lim \nmid \boldsymbol{x}_{[0,k]} \nmid_w / (k+1)$, where $\nmid u \nmid_w$ describes the number of occurences of w in u. we call $\alpha_{\boldsymbol{x}}$ the **slope** of $\boldsymbol{x}$ the number $\alpha_{\boldsymbol{x}} := f_1$ which is an **irrationnal number**.

Lemme 3 *if $\boldsymbol{x}$ is a sturmian sequence, and u and v two words of $\boldsymbol{x}$, of the same size then for all $n \geq 0 : | \nmid \boldsymbol{\tau}^n(u) \nmid_w - \nmid \boldsymbol{\tau}^n(v) \nmid_w | \leq 2^k |w|$*

Proof. For $n = 0$ it correspond to the property of balancedness of the sturmian sequences [**?**]. For $n > 0$ we can see that any preimages $w' \in \boldsymbol{\tau}^{-n}(w)$ and we have $| \nmid u \nmid_{w'} - \nmid v \nmid_{w'} | \leq |w'| \leq |w|$ thanks to the case $n = 0$. The following is a consequence of the triangular inequality and the fact that $Card(\boldsymbol{\tau}^{-n}(w)) \leq 2^k$.

Lemme 4 *If $\boldsymbol{x}$ is a sturmian sequence, then for all $n \geq 0$ and word w we have that $r_w(\boldsymbol{\tau}^k \boldsymbol{x}) \in \mathbb{Z} + \alpha\mathbb{Z}$, meaning that the slope remains an irrationnal number.*

Proof. For $n = 0$, it is a caracterisation of sturmian sequences ([**?**]). For $n > 0$ we have $r_w(\boldsymbol{\tau}^n(\boldsymbol{x})) = \sum\limits_{\tilde{w} \in \boldsymbol{\tau}^{-n}(w)} r_{\tilde{w}}(\boldsymbol{x})$.

Thanks to 4 we can conclude that $\mathrm{C}(\boldsymbol{\tau}^n(\boldsymbol{x}), \ell) \geq \ell + 1$ meaning that if $\boldsymbol{x}$ is sturmian, $\boldsymbol{\tau}(\boldsymbol{x})$ can not be periodic and can be sturmian. We will now show that $\boldsymbol{\tau}(\boldsymbol{x})$ will be **quasi sturmian** and how the complexity evolves through the automata iterations.

Theorem 5. $\forall n \geq 0, \exists L_n \geq 0 : \forall \ell \geq L_k$

$$\mathrm{C}(\boldsymbol{\tau}^n(\boldsymbol{x}), \ell) = \ell + n + 1 : P_n.$$

Proof. we reason by induction. It is easy to see that P_0 is true and let suppose that $(P_k)_{k \leq n}$ are true. Let now focus on $\boldsymbol{\tau}^n(\boldsymbol{x})$ and $\boldsymbol{\tau}^{n+1}(\boldsymbol{x})$.

$\forall i \in \mathbb{Z}$ we know that every word $\boldsymbol{\tau}^{n+1}(\boldsymbol{x})[i, i+\ell-1]$ of size ℓ is the image of a word $\boldsymbol{\tau}^n(\boldsymbol{x})[i, i+\ell]$ of size $\ell + 1$, i.e. $\boldsymbol{\tau}^n(\boldsymbol{x})[i, i+\ell] \in \boldsymbol{\tau}^{-1}(\boldsymbol{\tau}^{n+1}(\boldsymbol{x})[i, i+\ell-1])$. We know that

$$P_n \Rightarrow \forall \ell + 1 \geq L_n : \mathrm{C}(\boldsymbol{\tau}^n(\boldsymbol{x}), \ell + 1) = \ell + 1 + n + 1 = \ell + n + 2.$$

If $Card(\tau^{-1}(\tau^{n+1}(\boldsymbol{x})[i, i+\ell-1])) = 1 \Leftrightarrow \tau^{-1}$bijective then $C(\tau^{n+1}(\boldsymbol{x}), \ell) = C(\tau^{n}(\boldsymbol{x}), \ell+1) = \ell+n+2$ and P_{n+1} would be true. We reason by the absurd :

Let $w := \tau^{n+1}(\boldsymbol{x})[i, i+\ell-1]$ for $i \in \mathbb{Z}$. The only way to have two words v_0 and v_1 such that $v_0 \neq v_1$ and $\tau(v_0) = \tau(v_1) = w$ is to have $v_0 = \tau_0^{-1}(w)$ and $v_1 = \tau_1^{-1}(w)$. Thanks to 3, for $w = 1$ we get that $| \nmid v_0 \nmid_1 - \nmid v_1 \nmid_1 | \leq 2^n$ by noting that v_0 and v_1 are some words of lenght $\ell+1$ present in $\tau^n(\boldsymbol{x})$. Thanks to the construction of τ^{-1} we get also that

$$\nmid v_0 \nmid_1 + \nmid v_1 \nmid_1 = \ell+1 \Leftrightarrow \nmid v_1 \nmid_1 = \ell+1 - \nmid v_0 \nmid_1$$

cause $v_0 = v_1 + \mathbf{1}$. we get thanks to 2.2 that

$$\begin{aligned} & | \nmid v_0 \nmid_1 - (\ell+1 - \nmid v_0 \nmid_1)| \leq 2^n \\ \Leftrightarrow & |\ell+1-2 \nmid v_0 \nmid_1 | \leq 2^n \\ \Leftrightarrow & | \nmid v_0 \nmid_1 - \frac{\ell+1}{2}| \leq 2^{n-1} \Leftrightarrow |\frac{\nmid v_0 \nmid_1}{\ell+1} - \frac{1}{2}| \leq \frac{2^{n-1}}{\ell+1}. \end{aligned}$$

We also have that $| \nmid v_0 \nmid_1 - \nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1 | \leq 2^n \Leftrightarrow |\frac{\nmid v_0 \nmid_1}{\ell+1} - \frac{\nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1}{\ell+1}| \leq \frac{2^n}{\ell+1}$.

Finally we get that

$$\begin{aligned} & |\frac{1}{2} - \frac{\nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1}{\ell+1}| = |\frac{1}{2} - \frac{\nmid v_0 \nmid_1}{\ell+1} + \frac{\nmid v_0 \nmid_1}{\ell+1} - \frac{\nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1}{\ell+1}| \\ & \leq |\frac{1}{2} - \frac{\nmid v_0 \nmid_1}{\ell+1}| + |\frac{\nmid v_0 \nmid_1}{\ell+1} - \frac{\nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1}{\ell+1}| \leq \frac{2^{n-1}}{\ell+1} + \frac{2^{n-1}}{\ell+1} = \frac{2^n}{\ell+1} \\ & \overset{limit}{\Longrightarrow} \alpha_{\tau^n(\boldsymbol{x})} = \lim_{\ell \longrightarrow \infty} \frac{\nmid \tau^n(\boldsymbol{x})[0,\ell] \nmid_1}{\ell+1} = \frac{1}{2} \in \mathbb{Q} \end{aligned}$$

which is absurd thanks to 4.

2.3 Musical exemple

Let introduce two sequences, $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by their intervals and rythms, $\boldsymbol{x} = (r_{\boldsymbol{x}}, t_{\boldsymbol{x}}) = \big((4,-4)^{\mathbb{Z}}, \boldsymbol{t}_{\text{fib}} + \mathbf{1}\big)$ and $\boldsymbol{y} = (r_{\boldsymbol{y}}, t_{\boldsymbol{y}}) = \big((4,4)^{\mathbb{Z}}, \boldsymbol{t}_{\text{fib}}\big)$ we present an exemple where τ and Δ act both on rythms and intervals giving at the end the same result as $f_\Delta(4,4) = f_\tau(4,-4) = 0$ and $\tau(\boldsymbol{t}_{\text{fib}} + \mathbf{1}) = \tau(\boldsymbol{t}_{\text{fib}})$ and $\tau = \Delta$ in $\mathbb{Z}_2$. This exemple shows that Δ and τ can have a multi dimensional action on a music sequence.

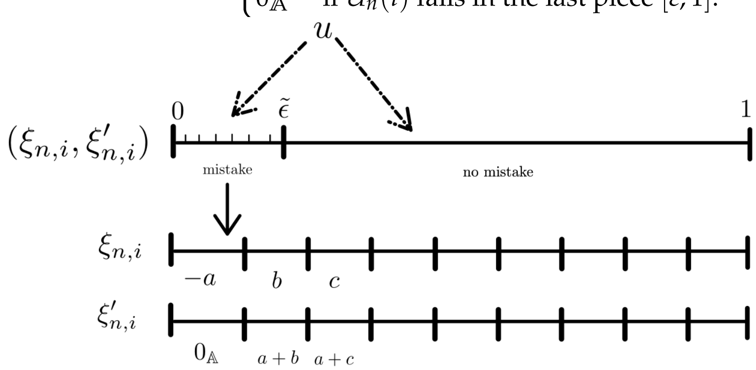
2.4 Real time joignings for Δ_ε and $\boldsymbol{\tau}_\varepsilon$

One idea is to use **real time joignings** (RTJ) inherited by the theory of classification of filtrations present in [1]. The main idea is to study two random music sequences following the dynamic of $F \in \{\Delta_\varepsilon, \boldsymbol{\tau}_\varepsilon\}$ $X = (X_{n,i})_{n,i\in\mathbb{Z}}$ and $X' = (X'_{n,i})_{n,i\in\mathbb{Z}}$ as explained in 1.1. We take $F = \boldsymbol{\tau}_\varepsilon$ to explicit how RTJ work.
For all $n \leq 0$, we introduce the random variable $Z_n := X'_n - X_n = \tilde{\Delta}(X_n, X'_n)$. The goal will be to have Z close to $\mathbf{0}$ meaning the two processes will be at a time really close despite the randomness and their initial difference. As $\boldsymbol{\tau}$ is a morphism on $\mathbb{A}^{\mathbb{Z}}$ it results that for all $n < 0$ $Z_n = \boldsymbol{\tau}(Z_{n-1}) + \xi_n - \xi'_n$. Until a time n_0 the two processes will be independant, $(X_n)_{n\leq n_0} \perp\!\!\!\perp (X'_n)_{n\leq n_0}$. Then we realise a **joignings** of their law of error described as following :

- The interval $[0,1]$ is cut into $|\mathbb{A}|+1$ sub-intervals, the first $|\mathbb{A}|$ being of length $\frac{\tilde{\varepsilon}}{|\mathbb{A}|}$, and the last of length $1-\tilde{\varepsilon}$. The $|\mathbb{A}|$ first pieces are called J_b, $b \in \mathbb{A}$, they correspond to the uniform measure part on $\mathbb{A}$ in the second way of writing the locator function f_{τ_ε}. For $u \in J_b$ we pose: $g'_{n,i}(u) := b$ and $g''_{n,i}(u) := b + a$.
- On the last piece, let $g'_{n,i}(u) := g''_{n,i}(u) := 0_{\mathbb{A}}$.

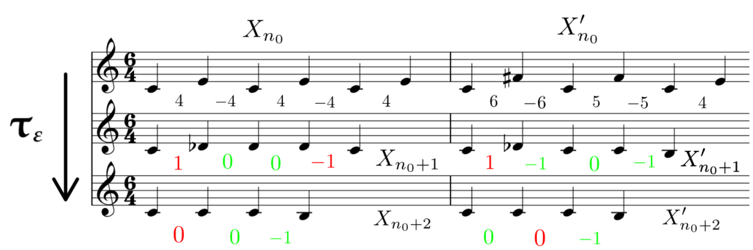
Thus, when we apply this strategy, we obtain

$$\xi'_n(i) - \xi_n(i) = \begin{cases} a & \text{if } U_n(i) \text{ falls in } [0,\tilde{\varepsilon}), \\ 0_{\mathbb{A}} & \text{if } U_n(i) \text{ falls in the last piece } [\tilde{\varepsilon}, 1]. \end{cases}$$



it has been established in [1] that with probability $1-\varepsilon$:

$$Z_0[-k,k] = (0_{\mathbb{A}}, \ldots, 0_{\mathbb{A}}) \iff X_0[-k,k] = X'_0[-k,k].$$

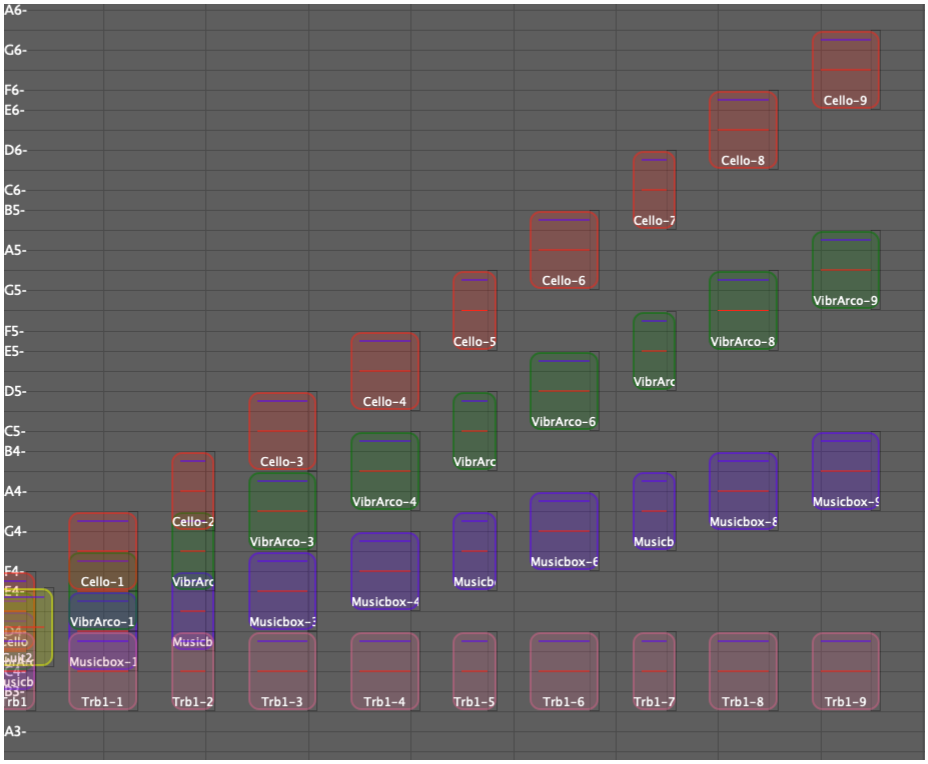


We finally get that the two sequences are the same in two iteration of $\boldsymbol{\tau}_\varepsilon$.

3 Implementation and new approach

UPISketch is a software which is the creation of the *Iannis Xenakis Center* based in Rouen, France. The software is born in 2017 and enable composers to draw musical elements and generate a freer music coming also from the mathematics. Its intention is to gather the best of both worlds : continuous and discrete and find bridges between both.

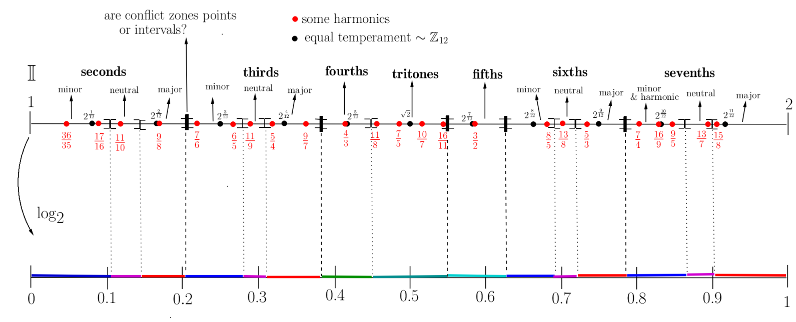
A module called **Automaton** has been created in UPISketch in order to implement the CA $\boldsymbol{\tau}$ and Δ on music sequences. We can select a finite or infinite sequence $\boldsymbol{x} = (r_i, T_i)_{i \in \mathbb{Z}}$ with r its intervals and T its rythms. r can be finite or periodic, T can be just simple linear rythms or a sturmian sequence. We can then select $F \in \{\Delta, \boldsymbol{\tau}\}$ and compute the iterations (*choosen line*) in the direction of the images or the preimages, the *column constraint* being which elements you select in order to go in the past. The *seeds* are the basic element of the substitution needed to generate a sturmian sequence in our case. We give in the next picture, an exemple in UPISkecth of different sequences giving the same result $\mathbf{0}^{\mathbb{Z}}$ by Δ as they are all monochromatic, i.e. made of one interval.



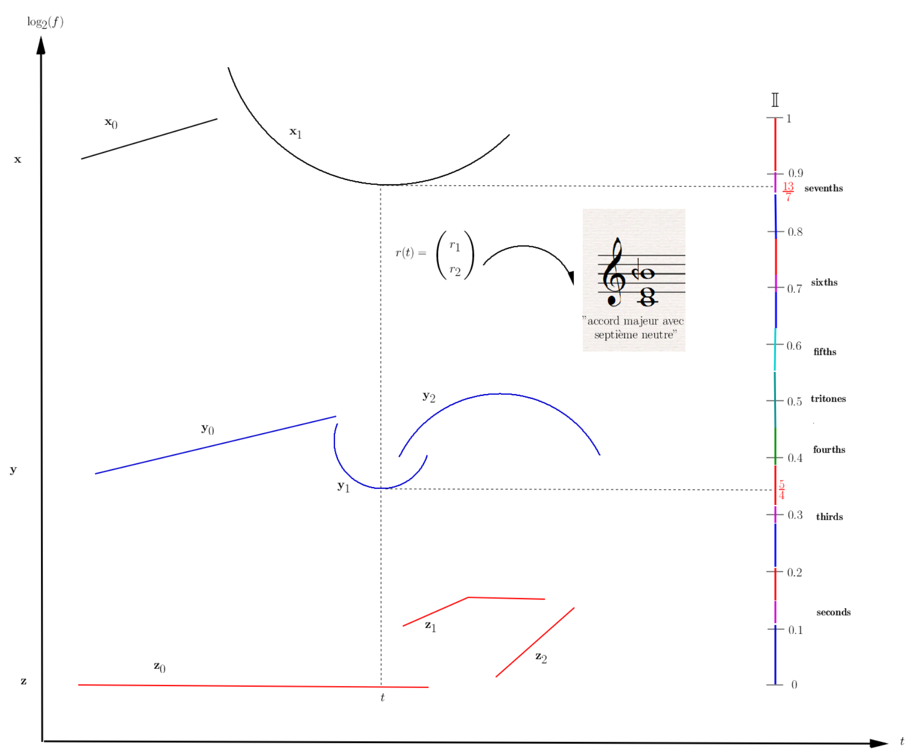
New approach In the software, we consider the sound events with their time intervals, their frequencies (or pitches), their timbres (waveform), their amplitudes, the space of configurations is thus a set of infinite sequences made by an alphabet $\mathbb{A}$ inspired by those dimensions. The classical notion of voice in music can be interpreted simply as a **music sequence**: $\boldsymbol{x} = (x_0, x_1, ..., x_n, ...)$ where $\boldsymbol{x}(i) := \boldsymbol{x}_i = (f_i, o_i, a_i, T_i)$ with

$$\overbrace{f_i}^{\text{the frequency}} \begin{array}{rcl} : T_i & \to & \mathbb{R}_+ \\ t & \mapsto & f_i(t) \end{array}, \quad \overbrace{o_i}^{\text{the timbre}} \begin{array}{rcl} : T_i & \to & \mathcal{C}(\mathbb{R}, \mathbb{R}) \\ t & \mapsto & o_i(t) \end{array} \text{ et } \quad \overbrace{a_i}^{\text{the amplitude}} \begin{array}{rcl} : T_i & \to & \mathcal{C}(\mathbb{R}, \mathbb{R}) \\ t & \mapsto & a_i(t) \end{array}$$

and T_i the time interval for the ith element, or also called **time support**. For chords, the frequencies f_i must be replaced by frequency vectors. In the case of a gesture, or simply said, a curve, it is a real function. The notion of a sequence is essential because it allows us to order the music and our perception of it in a discrete way and thus to fall easily into the cellular automata formalism. Instead of inputing the frequencies and time supports as mentionned previously, it is really common to work with intervals and rythms. A non avoidable question in music is to consider it in a discrete or a continuous way. For intervals we give an idea of how *pseudo discrete* way is to consider families of intervals :



It is fundamental to understand that we can choose and will specify the alphabet $\mathbb{A}$ according to the situation. The last image suggest that even if we get a result in $\mathbb{Z}_{12}$ or $\mathbb{Z}_7$ for instance, we can apply it in a continuous set and vice-versa.



To conclude this part we suggest a construction of τ and Δ in that specific case acting on the four dimensions of $\mathbb{A}$, taking x, y, z sequences in input as in the previous image. $\Delta(x) = x^{-} * \sigma(x)$ and $\tau(x) = x * \sigma(x)$ and let $\tilde{f} = \log_2(f)$, $o_i(t) = 0$ is interpreted as a simple sinusoïdal waveform and $a_i(t) = 0$ a medium intensity.

$$f_\Delta(\boldsymbol{x}_i, \boldsymbol{x}_{i+1})(t) = \begin{cases} \tilde{f}_{i+1}(t) - \tilde{f}_i(t) \\ o_{i+1}(t) - o_i(t) \\ a_{i+1}(t) - a_i(i) \\ \frac{T_i + T_{i+1}}{2} \end{cases} \text{ and } f_\tau(\boldsymbol{x}_i, \boldsymbol{x}_{i+1})(t) = \begin{cases} \tilde{f}_{i+1}(t) + \tilde{f}_i(t) \\ o_{i+1}(t) + o_i(t) \\ a_{i+1}(t) + a_i(i) \\ \frac{T_i + T_{i+1}}{2} \end{cases}$$

4 Conclusion & perpsectives

We saw that deterministic and probabilistic cellular automata are natural objects in music that can act on all the dimensions of a music sequence to generate new elements which are correlated to the original sequences . We explicited their use on periodic and sturmian sequences. We then presented the use of real time joignings as a way to mix determinism and probabilities. We also saw that thanks to UPISketch, it was a natural direction to explore the action of CA on a continuous alphabet or pseudo discrete.

It would be an interesting question to explore the case of the action of Δ and τ on episturmian sequences which is the generalisation of sturmian sequences in $\mathbb{Z}_N$ and see if the theorem on the linear evolution of complexity via the iteration of the automata stands. Episturmian would be then interpreted as interval sequences to code a melody. An other perspective is to explore new kind of automata like the bipermutative and expanding automata which are more complex that the sigma-polynomial automata, and find exemples that are natural in music.

As UPISkecth is working on the concept of page (frequency again times), 2-D automata could be also created taking in input a UPISkecth page and act in the same dynamic that in the game of life of Conway, a new page where cells would be the elements, and could be created with local interactions to describe the rules of evolution.

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