Parsimonious voice-leading in just intonation and hexatonic-like cycles

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Abstract. We present a context for voice-leading between chords whose successive intervals are expressed as ratios given by a sequence of positive integers, such as 4:5:6 for the just major triad in root position. We interpret the hexatonic cycle as a chord sequence of major and minor triads where at each step one positive integer changes by plus or minus one. This leads to a definition of parsimonious voice-leading in just intonation (JI) which is analogous to the notion of movement of one voice by one semitone in the equal-tempered (ET) context. Additionally, a graph of trichords with voice-leading is given in both the ET and JI contexts. A sequence of hexatonic-like cycles in JI is constructed, which forms a type of auditory hexatonic spiral.

Keywords: voice-leading · hexatonic cycle · just intonation.

1 Introduction and Background

The hexatonic cycle plays a central role in the notion of parsimonious voice-leading in the 18^{th} and 19^{th} century tonalities (see [1]). As a chord progression between major and minor triads, it cycles back to its origin and accomplishes this by the minimal voice-leading changes by one semitone in one voice at each step. In this way it can be seen as fitting naturally into the context of equal temperament (ET) where the semitone is an equal division of the octave into 12 parts and is the minimal interval for voice-leading. Although this type of structure is not technically feasible in just intonation (JI), it is straight-forward to construct approximations of the hexatonic cycle using intervals and chords in JI. We call such cycles, or chord progressions, "hexatonic-like". Further, one can ask if such cycles fit naturally into the JI context, and whether the voice-leading can be described in a simple, or *minimal* way.

Minimal (or parsimonious) voice-leading typically refers to small changes in pitch in one voice. In ET the smallest change in pitch is of course one semitone, and this is independent of starting pitch. In ET, the two nearest neighbors to any pitch are one semitone lower or higher. In JI, however, there is no smallest interval, allowing for endless types of intervals, scales, and harmonic structures. This variety can be seen as a strength of JI, yielding a wide range of expressivity, but it also seems to limit the structural aspects, such as minimal voice-leading, as exhibited in the hexatonic cycle. One of the main goals of this paper is to propose a type of minimal voice-leading in JI which also takes into account the complexity of the intervals used to form chords.

Another important aspect of JI is its relation to the physical concept of the harmonic series, connecting the notions of interval, scale and harmony to small integer ratios which are present in the harmonic tones generated by musical instruments. These aspects are explored in depth by Helmholtz in his classic book from 1885 [2], in which he discusses intervals (as fundamental frequency ratios) and scale systems as they have developed over the past two millenia. He also discusses basic harmonic progressions, but does not attempt any systematic treatment or classification. Harry Partch, in the 1940's, initiated another detailed analysis of JI, which is summarized in his book [3]. He proposes scale systems containing intervals which are rich in historical and numerical significance, and extends the usual collection of small integer ratios: 3/2 (perfect fifth), 4/3 (perfect fourth), 5/4 (major third), and so on, to include smaller intervals like the commas 81/80 (Syntonic) and $(3/2)^{12}/2^7$ (Pythagorean), which are each approximately 1/5 of an ET semitone. He also uses the notion of prime limit: limiting the largest prime factor of the positive integers used as frequency ratios. Partch emphasizes the use of 11-limit intervals, but does not include a detailed analysis of voice-leading.

Later authors, such as Kelley (in [4]), address parsimonious voice-leading in the context of JI with respect to the Tonnetz. Kelly's approach is to work with specific just intervals (ratios 3/2, 5/4, 6/5, 16/15, and 25/24) and to express all movements between triads with these. In order to achieve the cyclical nature of the tonnetz, he also considers various approaches to tempering of intervals and dealing with commas. In this sense, the transition of tuning systems from antiquity to modern times is reflected in his JI construction of the tonnetz.

In this paper we do not use prime limits and we do not restrict JI intervals to any size. We do, however, focus on approximation of the ET semitone by JI intervals in order to mimic the structure of the hexatonic cycle and its minimal voice-leading. We then exhibit an infinite family of "hexatonic-like" cycles in the space of trichords. We see this construction as an entry point into an exploration of harmony and voice-leading in JI. Since the hexatonic cycle has been noted for its "tonal ambiguity" in the context of ET, we feel that it is an appropriate place to consider the subtle changes in tonal ambiguity reflected in the small interval changes in the context of JI.

In section 2 we give a basic example which illustrates the central idea of minimal voice-leading in JI. This is followed by more general definitions of pitch and chord spaces in sections 3 and 4. In sections **??** and **??** we define hexatonic and hexatonic-like cycles. In sections 7, 8, and 9 we consider extended examples and future work.

2 A Motivating Example

Before making formal definitions regarding pitch spaces and voice-leading in JI, we give an example which sets the stage. We work out the example first with ET, then with JI. For simplicity, we use the terms *pitch* and *fundamental frequency* as somewhat interchangeable and without regard to issues of perception.

Suppose we play an A minor triad in root position centered at middle C on the piano, followed by the parallel A major triad in root position. Also assume closed position, so that the three pitches are within an octave. We can represent this simple chord progression as:

$$(A, C, E) \longrightarrow (A, C\sharp, E).$$

We also assume that the piano is tuned (approximately) in equal temperament (ET) with the A below middle C having fundamental frequency 220 Hz. With these assumptions, we have numerical values of fundamental frequency for these pitches as follows:

$$A\ :\ 220, \quad C\ :\ 261.63, \quad C\sharp\ :\ 277.18, \quad E\ :\ 329.63.$$

These are computed using the semitone multiplier (or frequency ratio) $2^{1/12}$:

$$220 \cdot 2^{3/12} = 261.63, \quad 220 \cdot 2^{4/12} = 277.18, \quad 220 \cdot 2^{7/12} = 329.63.$$

So we could also represent the progression as

$$(220, 261.63, 329.63) \longrightarrow (220, 277.18, 329.63)$$

or as

$$(220, 220 \cdot 2^{3/12}, 220 \cdot 2^{7/12}) \longrightarrow (220, 220 \cdot 2^{4/12}, 220 \cdot 2^{7/12}).$$

A simpler, and slightly more abstract, representation uses only a starting pitch as the lowest voice of the first chord, declared as integer value zero, then represents any other pitch as the number of semitones distant from the starting pitch with integer values. So with the understanding that A 220 Hz is our zero, we can represent the chord progression as:

$$(0, 3, 7) \longrightarrow (0, 4, 7).$$

Finally, if we are interested in expressing only the chord types in this progression, irrespective of starting pitch, we can use the representation which counts only the number of semitones separating voices in each chord. So the integer triple (a, b, c) reduces to $[x, y]$, with $x = b - a$ and $y = c - b$. Then the chord progression can be described as a particular realization of

$$[3, 4] \longrightarrow [4, 3],$$

which satisfies the two requirements: 1) the lowest pitch of the first chord is A 220 Hz, and 2) the progression to the second chord is done with minimal voice-leading. We take the second requirement to mean that any movement of voices is done with the smallest number of semitones. More precisely, using the triples of integers to describe the realization, one starts with $(0, 3, 7)$ and progresses to (a, b, c) while minimizing the sum of absolute differences in each voice:

$$|a - 0| + |b - 3| + |c - 7|.$$

In this case the minimum is trivially seen to be 1. Equivalently, moving from a major triad in root position to a minor triad in root position with minimal voice-leading is achieved by the parallel transformation P which raised the third of the triad by one semitone.

Next, we consider a realization of this chord progression in just intonation (JI). It is reasonable to still write the chord progression as

$$(A, C, E) \longrightarrow (A, C\sharp, E)$$

and to assume that A has fundamental frequency 220 Hz. The intervals are, however, flexible. We will choose the simplest frequency ratios, which minimize the *height* of the fraction $\frac{a}{b}$, or simply $max\{|a|, |b|\}$. This yields the values 5/4 for the major third, 6/5 for the minor third, and the resulting product 3/2 for the perfect fifth. These intervals can be compared to ET, to get some idea of their relative size. The cent value x of a frequency ratio R is defined so that $x(2^{1/12}) = 100$ cents:

$$x = 1200 \log_2(R).$$

Here are the cent values: $x(5/4) = 386.31$, $x(6/5) = 315.64$, $x(3/2) = 701.96$.

With these assumptions, we have numerical values of fundamental frequency for the pitches as follows:

$$A\ :\ 220, \quad C\ :\ 264, \quad C\sharp\ :\ 275, \quad E\ :\ 330.$$

These are computed using the chosen frequency ratios:

$$220 \cdot (6/5) = 264, \quad 220 \cdot (5/4) = 275, \quad 220 \cdot (3/2) = 330.$$

So we can represent this JI chord progression as

$$(220, 264, 330) \longrightarrow (220, 275, 330).$$

Next, we can represent the chord types without reference to the lowest pitch of the first chord. Here the two intervals are represented by a pair of frequency ratios to indicate the type of each of the two chords. Then the chord progression can be described as a particular realization of

$$[6/5, 5/4] \longrightarrow [5/4, 6/5],$$

which satisfies the requirement that the lowest pitch of the first chord is A 220 Hz. To understand the voice-leading which is induced by our choice of intervals, we can compute the change in the middle voice as a frequency ratio:

$$\frac{220 \cdot (5/4)}{220 \cdot (6/5)} = \frac{25}{24}.$$

To get some idea of the relative size of this voice-leading step, we compare to ET and see that the middle voice is raised by $x(25/24) = 70.67$ cents. This interval is a direct consequence of our choice of intervals in this realization. In order to see that this fits into a notion of *minimal voice-leading* in JI, we need another representation of the chord types as integer triples.

Let the integer triple (p, q, r) with $p < q < r$ be associated with the pair of frequency ratios q/p and r/q. Then clearly any multiple $k(p, q, r) = (kp, kq, kr)$, $k \neq 0$, is also associated to q/p and r/q. The condition implies that $q/p > 1$ and $r/q > 1$. So we can interpret these frequency ratios as describing the intervals of a trichord of pitches increasing left to right. Conversely, any pair of frequency ratios a/b and c/d, with $a > b$ and $c > d$, can be encoded in a triple of rational numbers $(1, a/b, ac/bd)$ which also corresponds to an integer triple $(p, q, r) = (bd, ad, ac)$ with $p < q < r$.

In our example the major triad has rational triple type $(1, 5/4, 3/2)$ or integer triple type $(4, 5, 6)$, and the minor triad has rational triple type $(1, 6/5, 3/2)$, or integer triple type $(10, 12, 15)$. If we fix a starting fundamental frequency of $f_0 = 220$, then we can form the two trichords in our example: $220(1, 5/4, 3/2) = (220, 275, 330)$ and $220(1, 6/5, 3/2) = (220, 264, 330)$. The chord progression can now also be described with the two integer triples, with the assumption that the lower voice in the first chord is $f_0 = 220$:

$$(10, 12, 15) \quad \longrightarrow \quad (4, 5, 6).$$

By the comments above, for any positive integers k and m we have the equivalent progression of chord types:

$$k(10, 12, 15) \quad \longrightarrow \quad m(4, 5, 6).$$

Now if we choose $k = 2$ and $m = 5$, we get the same progression in the form:

$$(20, 24, 30) \quad \longrightarrow \quad (20, 25, 30).$$

The middle coordinate can now be seen easily to change by one unit, or equivalently, it is multiplied by the ratio $\frac{25}{24}$.

We can see in this form that a concept of minimal voice-leading arises. In particular, we have a movement between two chord types expressed as integer triples, and the voice-leading can be characterized as “only one coordinate moves up or down by integer value one". This is analogous to the ET voice-leading where only one voice changes by one semitone. So we define minimal voice-leading between two chord types (p, q, r) and (p', q', r') to be one in which only one coordinate changes by value plus or minus one. To give a realization of such a chord progression one simply uses the associated rational types normalized to have a 1 in one of the coordinates that does not change. For example, if p changes but $q = q'$ and $r = r'$ are fixed, then use $(p/r, q/r, 1)$ and $(p'/r, q/r, 1)$ and multiply both triples through by some fundamental frequency f_0. This is our notion of minimal or parsimonious voice-leading in JI. It is dependent on representing trichord types by triples of integers which in turn express the intervals of the trichord as rational number frequency ratios. It is also dependent on the size of the integers present in the triples. The larger the integers, the smaller the size of the minimal voice-leading intervals associated to a particular triple (p, q, r). By minimal voice-leading interval we mean the absolute cent value of any one of the frequency ratios used to move one coordinate (or voice) of the triple up or down by one:

$$\frac{p-1}{p}, \frac{p+1}{p}, \frac{q-1}{q}, \frac{q+1}{q}, \frac{r-1}{r}, \frac{r+1}{r}.$$

Since $p < q < r$ the ratio $\frac{r+1}{r}$ will produce the smallest cent value.

In the following sections we explore this concept and how it is related to our construction of *hexatonic-like* cycles.

3 Pitch Spaces and Intervals in ET and JI

This kind of structure is less common in the context of just intonation (JI), which we define broadly in this paper to be the pitch space starting from any fixed fundamental frequency f_0 and then including all positive rational number multiples of f_0:

$$JI_{f_0} = \{\frac{a}{b} f_0 : a, b \in \mathbb{Z}_+\}.$$

By way of contrast, the pitch space of ET can be defined as all multiples of f_0 by integer powers of $2^{\frac{1}{12}}$:

$$ET_{f_0} = \{2^{\frac{k}{12}} f_0 : \ k \in \mathbb{Z}\}.$$

A few points about these definitions: The pitch space ET_{f_0} is an additive group isomorphic to the integers $\mathbb{Z}$, whereas the pitch space JI_{f_0} is a multiplicative group isomorphic to the group of positive rational numbers $\mathbb{Q}_+$. This group is a free Abelian group of countably infinite rank, generated multiplicatively by the prime numbers P:

$$JI_{f_0} = \{ \prod_{p \in S \subset P} p^{k_p} f_0 : k_p \in \mathbb{Z} \, |S| < \infty\}.$$

Thus, any fundamental frequency in ET_{f_0}, represented by some real number f_1 say, can be approximated arbitrarily closely by values in JI_{f_0}, but the converse is not true. Also, any positive real number f_1 can be approximated by a value in JI_{f_0}, for any fixed f_0. If we restrict these spaces to subsets of humanly audible fundamental frequencies, then in the case of ET we get a finite set, equal to the number of semitones in some range, say 20 Hz to 20 kHz, or about 120 values. In the case of JI, we get an infinite set, although many pitches are, to a human ear, practically indistinguishable, for example if they differ by less than one cent, or a ratio of $2^{\frac{1}{1200}}$.

We may also attempt to restrict these pitch spaces according to the complexity of intervals, or frequency ratios, with respect to the starting frequency f_0. In the case of ET there is no such restriction as each interval is specified explicitly by some integer k of semitones. In the case of JI, however, one can impose limits on the complexity of intervals in various ways, such as using height functions. The naive height function of a rational number $H(r)$ is simply the maximum of the absolute values of integers $|a|$ and $|b|$, where $r = a/b$ is in reduced form. The logarithmic height $h(r) = \log(H(r)$ can also be used. If one restricts $JI_{f_0}$to frequency ratios of bounded height $H(a/b) \leq N$, for some integer $N > 0$, then clearly this results in a finite collection of frequencies, with both smallest and largest intervals with respect to the frequency f_0. If we assume $a/b > 1$, we have cent values c satisfy:

$$0 < \log_2\left(\frac{N}{N-1}\right) \leq \frac{c}{1200} \leq \log_2(N).$$

4 Chord spaces and voice-leading in ET and JI

We define spaces of trichords in ET and JI literally as tuples of fundamental frequencies, and abstractly as integers or rational numbers which can be used to specify those fundamental frequencies. First, let

$$ET^3_{f_0} = \{(x, y, z) : x, y, z \in ET_{f_0}, x < y < z\},$$

and

$$JI^3_{f_0} = \{(x, y, z) : x, y, z \in JI_{f_0}, x < y < z\}.$$

Then, removing the dependence on f_0, we also define

$$ET^3_{\mathbb{Z}} = \{(a, b, c) \in \mathbb{Z}^3 : 0 < a < b < c)\},$$

and

$$JI^3_{\mathbb{Q}} = \{(\alpha, \beta, \gamma) \in \mathbb{Q}^3 : 0 < \alpha < \beta < \gamma)\}.$$

Next, we define trichord types in ET and JI by specifying the intervals between the lower and middle voices, and the middle and upper voices. In this way the structure of the chord is given without reference to the starting value or lower voice. In ET we simply specify the number of semitones between a and b as $x = b - a$, and between c and d as $y = c - b$, and define the semitone separation type, or s-type $[x, y]$ (as in [5]). The space of trichord types in ET is then defined as:

$$ET^3 = \{[x, y] : 1 \leq x, y \in \mathbb{Z}\}$$

Each s-type $[x, y]$ then determines a trichord type in ET up to transposition. The set $ET^3$of all s-types can be represented as a planar graph with nodes $[x, y]$, for

$x, y \geq 1$. We connect any two nodes with an edge to represent the existence of a parsimonious voice-leading between the two s-types. This produces edges from any node $[x, y]$ to the following four neighbors, as long as these are nodes in the graph: $[x-1, y]$, $[x+1, y]$, $[x, y-1]$, and $[x, y+1]$. If a triple of integers (a, b, c) is of s-type $[x, y]$, then the above nodes can be realized by the voice-leadings $a \to a+1$, $a \to a-1$, $c \to c-1$, and $c \to c+1$, respectively. We also obtain two more edges from $[x, y]$ to $[x-1, y+1]$ and $[x+1, y-1]$ which can be realized as voice-leadings $b \to b-1$ and $b \to b+1$ respectively. The boundary (where $x = 1$ or $y = 1$) is a special case. Away from the boundary every node $[x, y]$ has degree 6. Any movement between s-types can be realized as a path on this graph with length equal to the minimal number of edges traversed, or equivalently the minimal number of parsimonious transitions between trichords.

For example, consider the s-type $[4, 4]$ which represents the augmented triad. The six neighbors of this node are the s-types $[4, 3]$, $[3, 4]$, $[3, 5]$, $[4, 5]$, $[5, 4]$, and $[5, 3]$. These are the major and minor triads and their inversions, which we label as M, m, M_1, m_1, M_2, and m_2.

For JI, the intervals of a trichord can be specified as frequency ratios β/α and γ/β but it is somewhat simpler to clear denominators in the triple (α, β, γ) and represent the trichord type as a triple of integers (p, q, r). We define the *j-types*, or trichord types, in JI as:

$$JI^3 = \{(p, q, r) : 0 < p < q < r \in \mathbb{Z}\}$$

There are many such triples (kp, kq, kr), for any positive integer k, which produce the same ratios, but we prefer to distinguish these as separate j-types in the set JI^3 since each one will have a different minimal voice-leading neighborhood. One can also think of these sets of triples as lying on the line through the origin in $\mathbb{Z}^3$ containing (p, q, r), or as points in the projective plane $\mathbb{P}^2(\mathbb{Z})$. We will return to this point of view in the last section.

This choice of j-type matches the historical representation of just trichords such as the major triad in root position by the ratios 4:5:6, which of course implies the two frequency ratios 5/4 and 6/5 for the just major third and just minor third. If we reverse these intervals we get 10:12:15 indicating a just minor triad. (We will see in section 6 that there are many other interesting versions of these triads in JI^3.)

Next, we define minimal voice-leading in JI in a similar way as in ET, namely the smallest step from a triple $(p, q, r) \in JI^3$ is to add or subtract one on any coordinate. Since we work strictly with integer triples, this means that from the j-type $(4, 5, 6)$ we cannot move the middle voice, so the only minimal voice-leadings are to $(3, 5, 6)$ or $(4, 5, 7)$. To connect these voice-leadings to familiar intervals in ET it is useful to write down the cent values, and also the closest s-type $[x, y]$ as in table 1.

Table 1. minimal voice-leading from $(4, 5, 6)$

j-type	ratio	cents	ratio	cents	s-type
$(3, 5, 6)$	5/3	884.4	6/5	315.6	[9,3]
$(4, 5, 6)$	5/4	386.3	6/5	315.6	[4,3]
$(4, 5, 7)$	5/4	386.3	7/5	582.5	[4,6]

For the minor triad j-type $(10, 12, 15)$ we have slightly larger integers, with more possible voice-leadings, as displayed in table 2. Note that the second entry $(9, 12, 15)$ has the same chord type as $(3, 4, 5)$, which is a second inversion major triad. In this case, the minimal voice-leading in JI from $(9, 12, 15)$ to the $(10, 12, 15)$ minor triad is to increase the first voice by the just (minor) whole tone with ratio 10/9 and cent value 182.4. Similarly the last entry $(10, 12, 14)$ has the same chord type as $(5, 6, 7)$, which is a just diminished triad. The voice-leading from $(10, 12, 15)$ to $(10, 12, 14)$ is done by moving the upper voice down by the just semitone 15/14 with cent value 119.4. (Although we are not discussing seventh chords in this paper, it is worth noting that this diminished triad forms the upper three voices of the 4:5:6:7 harmonic seventh chord.)

Table 2. minimal voice-leading from $(10, 12, 15)$

j-type	ratio	cents	ratio	cents	s-type
$(10, 12, 15)$	6/5	315.6	5/4	386.3	[3,4]
$(9, 12, 15)$	4/3	498.0	5/4	386.3	[5,4]
$(11, 12, 15)$	12/11	150.6	5/4	386.3	[2,4]
$(10, 11, 15)$	11/10	165.0	15/11	537.0	[2,5]
$(10, 13, 15)$	13/10	454.2	15/13	247.7	[5,2]
$(10, 12, 14)$	6/5	315.6	7/6	266.9	[3,3]

We summarize the above in the following:

Parsimonious voice-leading of trichords in JI^3 *We define parsimonious (or minimal) voice-leading for trichords in* JI^3 *to be movement of j-type* (p, q, r) *by integer value* 1 *or* -1 *in any coordinate. The interval size of such voice-leading expressed as a ratio* R *satisfies* $\frac{r+1}{r} \leq R \leq \frac{p}{p-1}$.

We call a j-type (p, q, r) in JI^3 *reduced* if $\gcd(p, q, r) = 1$. Note that multiples of a reduced j-type (p, q, r), for positive integers k produce j-types with the same chord structure, but different minimal voice-leading. Also, as k increases, the minimal voice-leading for (kp, kq, kr) decreases in size of intervals measured as frequency ratios or cent values. In order to allow for flexible voice-leading in JI between different literal trichords in $JI^3_{f_0}$ we consider the substitution of scaling (p, q, r) by the factor k to (kp, kq, kr) to have no effect other than to prepare for a different voice-leading. To be more precise, we observe that movement from j-type (p, q, r) to any other j-type (p', q', r') can be done through a path consisting of minimal voice-leading steps, or scaling. We define the *voice-leading distance* between two such j-types to be the minimal number of voice-leading steps in such a path, ignoring scaling. We define the *cent value distance* between two such j-types to be sum of the absolute cent values for each step in the shortest voice-leading path. For example, this corresponds to the ET case of minimal voice-leading between two trichords (changing one voice by one semitone) having cent value difference of exactly 100 cents. We illustrate these definitions with an example of the transition between two reduced j-types:

$$(3, 5, 7) \longrightarrow (8, 11, 14).$$

The naive voice-leading path would add 5, 6, and 7 to the voices p, q, and r, for a total of 18 minimal voice-leading steps. The shortest path, however, can be seen to require only two voice-leading steps, with a scaling in the middle:

$$(3, 5, 7) \longrightarrow (4, 5, 7) \sim (8, 10, 14) \longrightarrow (8, 11, 14).$$

The cent value changes in the chord transitions are 498.0 cents for the 4/3 perfect fourth in the first voice, and 165.0 cents for the 11/10 ratio in the second voice.

5 Hexatonic cycles in ET

We consider the hexatonic cycle as a subset of ET^3 consisting of the six nodes which are three major and three minor triads, with the following s-types:

$$M: \ [4, 3], \ m: \ [3, 4], \ M_1: \ [3, 5], \ m_1: \ [4, 5], M_2: \ [5, 4], \ m2: \ [5, 3].$$

As we travel around the cycle, nodes alternate between major and minor. The voice-leading along each edge of the graph consists of two types of transformation: P (parallel) and L (leading tone), and these also alternate. All of these comments are irrespective of starting node or direction around the cycle.

We use the subscripts 1 and 2 to indicate first and second inversions. In this triangular grid we display nodes according to s-type $[x, y]$ with $3 \leq x, y \leq 6$, and $x + y \leq 9$. The hexatonic cycle goes around the $+$ node which is the augmented triad, moving for example through the nodes:

$$M \longrightarrow m \longrightarrow M_1 \longrightarrow m_1 \longrightarrow M_2 \longrightarrow m_2 \longrightarrow M$$

The hexatonic cycle can be embedded in a cube inside $ET^3_{\mathbb{Z}}$, where all vertices now represent triples (a, b, c) of integers. For example, if the major triad in root position M is now $(0, 4, 7)$, then the vertices can be labeled as in figure 1.

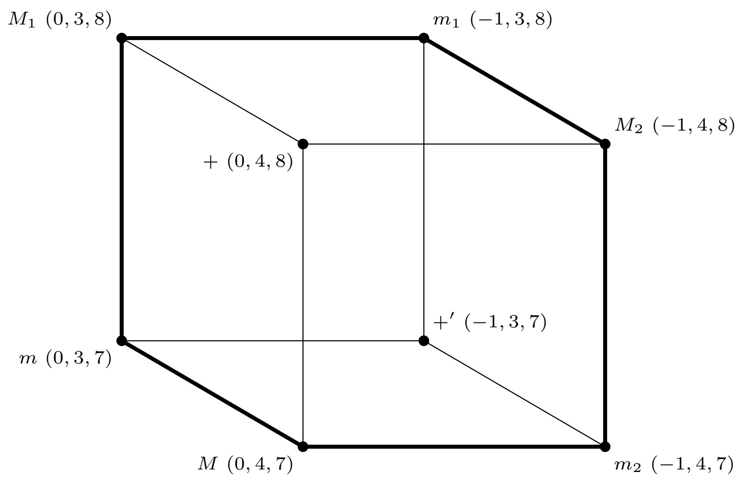


Fig. 1. Cube ET Hexatonic Cycle

Note that the augmented triad labelled $+'$ is the result of transposing $+$ down by one semitone, and that the path

$$+' \ (-1, 3, 7) \longrightarrow m \ (0, 3, 7) \longrightarrow M \ (0, 4, 7) \longrightarrow + \ (0, 4, 8)$$

moves by raising a then b then c, each by one semitone. So if this path is repeated we reach another augmented triad $+''$ $(1, 5, 9)$. Continuing in this way, we obtain a sequence of augmented triads, each surrounded by a hexatonic cycle on the remaining vertices of a cube. One obtains three of the major triads (and their parallel minor triads) at each stage, and four such cubes give us all twelve keys. These are illustrated in what is known as the “cube dance” of Douthett and Steinbach (see [6]).

6 Hexatonic-like cycles in JI

In order to explore possible versions of the hexatonic cycle in JI, we set a few goals:

1. use parsimonious (minimal) voice-leading in the JI context
2. find a cycle in JI^3 which matches the ET case above.
3. use small integer values in j-types (p, q, r)
4. see what other cycles of similar form in JI^3 arise from this construction.

Since we have observed that parsimonious voice-leading in JI^3 implies smaller intervals as the integers p, q, and r grow larger, we should begin by looking in the range where such intervals are close to an ET semitone. We start by considering the most well-known j-type $(4, 5, 6)$ that represents a major triad M, and consider multiples $(4k, 5k, 6k)$ where k is a positive integer. With $k = 1$ we are already stuck at the first step of moving M to m. The ET version requires moving the middle voice from s-type $[4, 3]$ to $[3, 4]$, which would correspond to moving the middle value q from 5 to 4, but then our first interval collapses to a unison.

For $k \geq 2$ we keep in mind the goal to work with transitions that are close to semitones. Here it helps to have some examples of ratios of type $(p + 1)/p$ approximately equal to 100 cents. We can see in table 3 that the closest such approximation occurs for $p = 17$. This suggests that we try $k = 4$. In this case, we can translate almost directly from the voice-leading in the ET case to JI^3, starting with the node $(16, 20, 24)$ as M. Lowering the middle value by one, is then equivalent to changing the middle voice of a trichord of this j-type by the frequency ratio 19/20 with cent value -88.8. The resulting cent values for the two intervals implied by $(16, 19, 24)$ are then 297.5 (19/16) and 404.4 (24/19), very close to an ET minor triad. This progression can be completed to give a nice approximation of the hexatonic cycle in JI as indicated in table 4. Note that each transition of (p, q, r) moves in the minimal way.

Table 3. just semitones

just ratio	15/14	16/15	17/16	18/17	19/18	20/19	21/20	22/21
cent value	119.4	111.7	105.0	99.0	93.6	88.8	84.5	80.5

We refer to the resulting cycle as a just hexatonic-like cycle, or JHC. We indicate the j-type (p, q, r) and multiplier k, as well as the starting node, in this case M, with the label $JHC(M = (4, 5, 6), k = 4)$.

Each line in table 4 gives the j-type as p, q and r, then the frequency ratio followed by cent value of each of the intervals. The ratios are reduced to emphasize the presence of familiar ratios like 5/4, 6/5, and 4/3. The s-type is also given so that we can see how close this JI version is to the ET version. As a rule of thumb, we consider a deviation of more than 50 cents from the ET version to have departed to a different tonality. We also use the same rule in displaying the s-type $[x, y]$, where x and y are the nearest integer to the cent values computed from the frequency ratios q/p and r/q, respectively. The largest deviation from ET in this JI version is the 25/19 perfect fourth which is almost 25 cents flat to the ET perfect fourth.

For $k = 3$ we find the only other case in which the hexatonic-like cycle satifies the rule of thumb with deviations from ET being less than 50 cents. In this case

Table 4. just hexatonic-like cycle: $JHC(M = (4, 5, 6), k = 4)$

chord	p	q	r	ratio	cents	ratio	cents	s-type
M	16	20	24	5/4	386.3	6/5	315.6	[4,3]
m	16	19	24	19/16	297.5	24/19	404.4	[3,4]
M_1	16	19	25	19/16	297.5	25/19	475.1	[3,5]
m_1	15	19	25	19/15	409.2	25/19	475.1	[4,5]
M_2	15	20	25	4/3	498.0	5/4	386.3	[5,4]
m_2	15	20	24	4/3	498.0	6/5	315.6	[5,3]

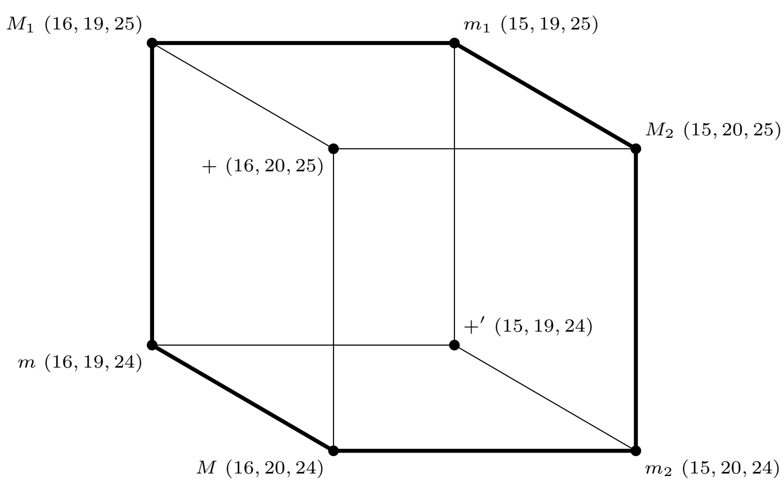


Fig. 2. Just Hexatonic-like Cycle

the maximum deviation from ET is the 15/11 perfect fourth of 537 cents, or 37 cents sharp to ET. We display this hexatonic-like cycle in table 5.

Table 5. just hexatonic-like cycle $JHC(M = (4, 5, 6), k = 3)$

chord	p	q	r	ratio	cents	ratio	cents	s-type
M	12	15	18	5/4	386.3	6/5	315.6	[4,3]
m	12	14	18	7/6	266.9	9/7	435.1	[3,4]
M_1	12	14	19	7/6	266.9	19/14	528.7	[3,5]
m_1	11	14	19	14/11	417.9	19/14	528.7	[4,5]
M_2	11	15	19	15/11	537.0	19/15	409.2	[5,4]
m_2	11	15	18	15/11	537.0	6/5	315.6	[5,3]

Note that these cycles are biased by the choice of starting with M. We will display a sequence of such JHC's which all start with M and use different voice-leadings. This is typical of JI constructions of intervals, chords and scales as well. There are many varieties. Typically, this plethora of choices does not lead easily to cyclical properties, but in this case at least we do see that we have a harmonic progression which returns to it's initial point exactly and does so in a pitch-preserving way.

For $k = 2$ or $k > 4$, we find that the largest deviation from ET is more than 50 cents away from ET in at least one of the intervals. We continue to use the abbreviation JHC for these cycles, and since they are built out of similar voice-leading from the starting point M, we continue to refer to them as "hexatonic-like" cycles.

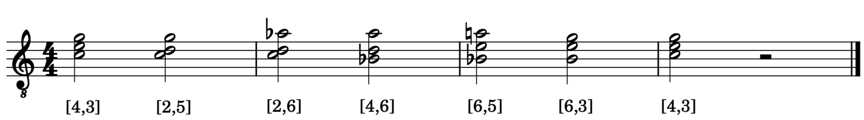
The case $k = 2$ is distinguished as the one with the largest intervals of minimal voice-leading, since the ratios are now for example 9/8 and 10/9, the JI major and minor whole tones. We display the data for this cycle in table 6. The hexatonic-like cycle now takes on a completely different character including tritones 10/7 of 617.5 cents, and 13/9 of 636.6 cents. Also, in this case we rename the last interval as o_2 since the closest s-type $[6, 3]$ matches the second inversion diminished triad. Note that this progression moves in whole steps or half steps according to the s-types. The first two chords are very close to their ET counterparts $[4, 3]$ and $[2, 5]$, and the next one has the wide tritone of 636.6 cents. This is followed by lowering the first voice by a wide whole tone resulting in the j-type $(7, 9, 13)$ which is approximated by s-type $[4, 6]$. But this is deceptive since the outer interval of this chord would be a minor seventh in ET but we now have a spread of 13/7 which is about 1071.7 cents, closer to a major seventh. At the next step we raise the middle voice by the just minor whole tone 10/9, resulting in the approximate s-type [6, 5]. A natural question to ask is what type of voice-leading is implied by this sequence in ET?

Table 6. just hexatonic-like cycle $JHC(M = (4, 5, 6), k = 2)$

chord	p	q	r	ratio	cents	ratio	cents	s-type
M	8	10	12	5/4	386.3	6/5	315.6	[4,3]
(m)	8	9	12	9/8	203.9	4/3	498.0	[2,5]
(M_1)	8	9	13	9/8	203.9	13/9	636.6	[2,6]
(m_1)	7	9	13	9/7	435.1	13/9	636.6	[4,6]
(M_2)	7	10	13	10/7	617.5	13/10	454.2	[6,5]
o_2	7	10	12	10/7	617.5	6/5	315.6	[6,3]

We suggest three possible realizations in the following figures. In each of them the first two chords are easily determined, since the ET and JI versions are so close. In fact, the intervals in the [2, 5] chord are closer to ET than even the initial major triad is. Next, we also have natural choices for the [2, 6] and [4, 6] chords, but the latter has the ambiguous property mentioned above, having spread of closer to a major seventh in the outer voices. If we stick with these types (which we do in the first two versions) then we have so far moved the lower two voices by a whole step and the upper voice by a half step. At this point, the first two versions diverge. In the first, in order to achieve the [6, 5] chord we need to adopt a voice-leading step of a whole tone in the upper voice and a half step in the middle voice. A compromise in version 2 instead moves only the middle voice by a whole step, arriving at a [6, 4] chord. This maintains voice-leading consistency (only one voice moves) at each step, but the realization is not as close to the original JI example. Either of these versions move to the [6, 3] by whole step or half step.

Fig. 3. ET realization of JHC($k = 2$) version 1



The third version diverges already at the third chord, raising the upper voice by a whole step, instead of a half step to reach the [2, 7] chord. This facilitates reaching the [6, 5] chord with consistent voice-leading, moving the first, then

Fig. 4. ET realization of JHC($k = 2$) version 2

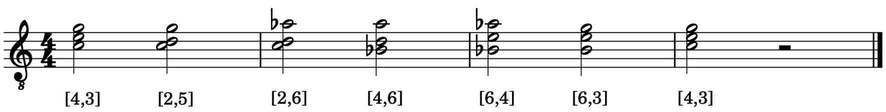
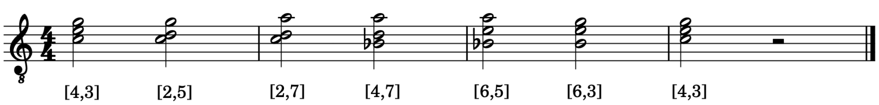


Fig. 5. ET realization of JHC($k = 2$) version 3



the second, voices by whole steps. In summary, although the first version is the closest to the original JI version in terms of the sizes of intervals, it is farthest from that version in terms of voice-leading. Also, version 2 moves the lower and middle voices by whole steps and the upper voice by half steps, whereas version 3 moves all voices by whole steps.

This example serves to show that minimal voice-leading in JI can have multiple interpretations in ET, each with different characteristics.

The case $k = 5$ is notable in that it is the only one which proceeds from M to m by swapping the place of the simplest JI intervals representing the major and minor thirds, namely 5/4 and 6/5, which are also inversions with respect to the just perfect fifth 3/2. These occur in table 7 as j-types $(20, 25, 30)$ and $(20, 24, 30)$. This is followed by the movement to $(20, 24, 31)$, raising the upper voice by the JI interval 31/30 or about 57 cents. Since the just major third 5/4 is already quite flat to ET at 386.3 cents, we arrive at 443.1 cents for the second interval, no longer within 50 cents of the the ET perfect fourth. So we have arrived at a different tonality, no longer qualifying as s-type $[3, 5]$, but closer to $[3, 4]$. Similarly, the next chord is no longer of closest s-type $[4, 5]$, and is marked $[4, 4]$.

Similar comments apply to the case $k = 6$, which has the same sequence of s-types as for $k = 5$. As we increase k, we expect to see the minimal voice-leading intervals decreasing, and hence the intervals present in the chords changing by smaller amounts. Each voice experiences a lowering or raising by steps which are one of the following three frequency ratios:

$$p/(p-1) \quad \text{or} \quad q/(q-1) \quad \text{or} \quad (r+1)/r.$$

Table 7. just hexatonic-like cycle $JHC(M = (4,5,6), k = 5)$

chord	p	q	r	ratio	cents	ratio	cents	s-type
M	20	25	30	5/4	386.3	6/5	315.6	[4,3]
m	20	24	30	6/5	315.6	5/4	386.3	[3,4]
m	20	24	31	6/5	315.6	31/24	443.1	[3,4]
$+$	19	24	31	24/19	404.4	31/24	443.1	[4,4]
M_2	19	25	31	25/19	475.1	31/25	372.4	[5,4]
m_2	19	25	30	25/19	475.1	6/5	315.6	[5,3]

In table 8 each of these ratios are computed in cents for various values of k. We use these cycles for increasing k in section 7 to construct a sequence of cycles with gradually diminishing voice-leading, which we call hexatonic spirals.

There are many other ways to construct just hexatonic-like cycles from some chosen starting point (p, q, r) which represents one of the six nodes in the hexatonic cycle. If we follow the rule of thumb that this starting node should be a JI realization of the chosen chord which differs in its defining intervals at most by 50 cents, and that the voice-leading should be minimal and should lead to the other nodes that also satisfy this rule of thumb, then there should be only finitely many such just hexatonic-like cycles.

7 The just hexatonic spiral

Although we have defined many JI chord progressions which are cyclical through a process of minimal voice-leading in JI, we note now that they form sequences of cycles with limiting properties, more similar to spirals. For example, the just hexatonic-like cycles $JHC(M = (4,5,6), k)$ for $k = 2, 3, 4, \ldots$ form a sequence, with dimishing voice-leading interval sizes. This sequence has the limiting cycle for large k which has voice-leading size that approaches zero. The resulting chords will then stabilize at M. From a psychoacoustic perspective, this will of course happen in a finite number of iterations. One can compute, for instance, that the value $k = 436$ is the first k for which the voice-leading in the cycle $JHC(M = (4,5,6), k = 436)$ consists of all steps below the value of 1 cent, or one percent of one semitone. We display in table 6 the first four iterations of this

Table 8. minimal ratios in $JHC(M = (4, 5, 6), k)$

k	p	q	r	$p/(p-1)$	$q/(q-1)$	$(r+1)/r$
3	12	15	18	150.6	119.4	93.6
4	16	20	24	111.7	88.8	70.7
5	20	25	30	88.8	70.7	56.8
6	24	30	36	73.7	58.7	47.4
7	28	35	42	63.0	50.2	40.7
8	32	40	48	55.0	43.8	35.7
9	36	45	54	48.8	38.9	31.8
10	40	50	60	43.8	35.0	28.6

Table 9. just hexatonic-like cycle $JHC(M = (4, 5, 6), k = 6)$

chord	p	q	r	ratio	cents	ratio	cents	s-type
M	24	30	36	5/4	386.3	6/5	315.6	[4,3]
m	24	29	36	29/24	327.6	36/29	374.3	[3,4]
(M_1)	24	29	37	29/24	327.6	37/29	421.8	[3,4]
(m_1)	23	29	37	29/23	401.3	37/29	421.8	[4,4]
M_2	23	30	37	30/23	460.0	37/30	363.1	[5,4]
m_2	23	30	36	30/23	460.0	6/5	315.6	[5,3]

Fig. 6. Just hexatonic spiral

$$(8, 10, 12) \to (7, 10, 12) \to (7, 10, 13) \to (7, 9, 13) \to (8, 9, 13) \to (8, 9, 12) \to$$
$$(12, 15, 18) \to (11, 15, 18) \to (11, 15, 19) \to (11, 14, 19) \to (12, 14, 19) \to (12, 14, 18) \to$$
$$(16, 20, 24) \to (15, 20, 24) \to (15, 20, 25) \to (15, 19, 25) \to (16, 19, 25) \to (16, 19, 24) \to$$
$$(20, 25, 30) \to (19, 25, 30) \to (19, 25, 31) \to (19, 24, 31) \to (20, 24, 31) \to (20, 24, 30) \to$$

sequence of cycles. At the end of each line we return to the M chord $(4k, 5k, 6k)$ but then proceed with smaller voice leading as k increases.

This notion of spiral is not meant to be visualized in the space of j-types, which can of course be embedded in 3-space. In fact, each cube in JI^3 containing one such cycle has one corner anchored at a point $(4k, 5k, 6k)$. So these cubes are all disjoint and of the same naive size. But they differ in the size of voice-leading. The point is that the voice-leading gets progressively tighter around the initial and repeating $(4, 5, 6)$ triad, so in terms of pitch we are spiraling inward toward this one chord.

8 Non-minimal voice-leading hexatonic-like cycles

We note here that it is certainly possible to abandon minimal voice-leading and construct interesting variations on this theme in JI. For example, still using the basic $(4, 5, 6)$ major triad with multiple $k = 25$ we construct a just hexatonic cycle from starting point or j-type $(100, 125, 150)$. To approximate semitone movement, we use step sizes $s_a = 6$, $s_b = 7$, and $s_c = 8$ on voices a, b, and c. This gives semitone ratio 125/118 in the middle voice of 99.8 cents, or almost exactly 100 cents. The other two voices use semitones 100/94 of 107.1 cents, and 158/150 of 90.0 cents. The hexatonic cycle then produces the table 10. Instead of a cube in JI^3 we now have a cycle which travels around a rectangular box with sides of lengths s_a, s_b, and s_c.

Table 10. $JHC(M = (4, 5, 6), k = 25, s_a = 6, s_b = 7, s_c = 8)$

chord	a	b	c	ratio	cents	ratio	cents	s-type
M	100	125	150	5/4	386.3	6/5	315.6	[4,3]
m	100	118	150	59/50	286.5	75/59	415.4	[3,4]
M_1	100	118	158	59/50	286.5	79/59	505.4	[3,5]
m_1	94	118	158	59/47	393.7	79/59	505.4	[4,5]
M_2	94	125	158	125/94	493.4	158/125	405.6	[5,4]
m_2	94	125	150	125/94	493.4	6/5	315.6	[5,3]

9 Further work

A natural extension of this work could be to consider tetrachords and seventh chords using the same principle of voice-leading in JI. Another direction to explore is the expansion of JI^3 to larger sets of integer triples (p, q, r). A first step in this direction is to relax the requirement $p < q < r$. This could be useful in considering the case where voices overlap and cross over other voices in the space of chord progressions. This could have meaning in the musical context of voices being implemented by different timbres, for example. A coordinate value of zero could indicate that a voice is silent. In addition to voice-leading, which uses the notion of closeness of triples (p, q, r), one could consider collections of chords as j-types which satisfy some algebraic conditions. This could be done with affine coordinates $(x, y, 1)$ and rational numbers x, y, which then correspond to projective integer coordinates (p, q, r). Allowing for negative coordinates could be done for algebraic completeness, but with the interpretation as frequency ratios by taking absolute values of coordinates. In this context one can consider integer points in the projective plane and its subsets. Of particular interest could be algebraic curves with group structure, such as elliptic curves. The notion of cycle comes up in this context since an elliptic curve may have a finite cyclic subgroup consisting of integer points, which can be interpreted as j-types of chords in JI with a natural ordering induced by the group action.

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